On p-adic Dedekind-Rademacher sums attached to Dirichlet characters

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Abstract The purpose of this paper is to generalize the author's preceding work on the construction of a p-adic analytic function interpolating the Dedekind sums attached to Dirichlet characters and the calculation of the radius of convergence of the function We define the generalized Dedekind-Rademacher sums by making use of Dirichlet characters and deduce an expression of the sums by the generalized Euler numbers. Applying the expression, we construct a p-adic analytic function interpolating the generalized Dedekind-Rademacher sums. The function is expressed as a linear combination of some p-adic functions interpolating the Euler numbers. The main result is the explicit expression of the radius of convergence of the function. Except for some special cases, the result is an analogue to the one for the Kubota-Leopoldt p-adic L-function, which interpolates the generalized Bernoulli numbers p-adically and plays an important role in the Iwasawa theory for cyclotomic fields.

Keywords [p-adic interpolation, Dedekind sums, Dirichlet character]

1 Introduction

For any real number x, we denote by $[x]$ the greatest integer not exceeding x, put $\{x\} = x - [x]$ and define

$$((x)) = \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \text{ is not an integer.} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

For positive integers h and k, the classical Dedekind sum $s(h,k)$ is defined by

$$s(h,k) = \sum_{\lambda \bmod k} \left(\left(\frac{\lambda}{k}\right)\right)\left(\left(\frac{h\lambda}{k}\right)\right). \tag{1}$$

The sum first appeared in Dedekind's study on the transformation properties of the η-function ($\eta(z) = e^{\pi iz/12}\prod_{n\geq 1}(1-e^{2\pi inz})$) under the modular group and in the case of $\gcd\{h,k\} = 1$ Dedekind showed the following reciprocity formula[1)]:

$$12hk\{s(h,k) + s(k,h)\} = h^2 - 3hk + k^2 + 1. \tag{2}$$

Generalizations of Dedekind sums and their reciprocity formulas have been studied extensively with many methods.

For each non-negative integer n, let B_n and $B_n(X)$ be the nth Bernoulli number and polynomial respectively, and define

$$\tilde{B}_n(x) = B_n(\{x\}).$$

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As a generalization of $s(h,k)$, Apostol defined the nth higher-order Dedekind sum as

$$s_n(h,k) = \sum_{\lambda \bmod k} \tilde{B}_1\Big(\frac{\lambda}{k}\Big)\tilde{B}_n\Big(\frac{h\lambda}{k}\Big) \tag{3}$$

and he generalized the formula (2) as

$$(n+1)\left(hk^n s_n(h,k) + h^n k s_n(k,h)\right) = nB_{n+1} + \sum_{j=0}^{n+1}\binom{n+1}{j}(-k)^{n+1-j}h^j B_{n+1-j}B_j \tag{4}$$

for positive integers h and k with $\gcd\{h,\ k\} = 1$ and odd n.

As a natural generalization of (3), we can define

$$s_{m,n}(h,k) = \sum_{\lambda \bmod k} \tilde{B}_m\Big(\frac{\lambda}{k}\Big)\tilde{B}_n\Big(\frac{h\lambda}{k}\Big) \tag{5}$$

for non-negative integers m and n. Further for real numbers α and β, we extend the sum (5) as

$$S_{m,n}\begin{pmatrix} h & k \\ \alpha & \beta \end{pmatrix} = \sum_{\lambda \bmod k} \tilde{B}_m\left(\frac{\lambda+\beta}{k}\right)\tilde{B}_n\left(\frac{h(\lambda+\beta)}{k} - \alpha\right), \tag{6}$$

which is often called the Dedekind-Rademacher sum. The reciprocity formula for (6) is studied by Rademacher and Carlitz[3∼5].

In addition to the reciprocity formulas, Rosen and Snyder constructed a p-adic interpolating function for the sums (3)[6]. The function is an analogue of the well known Kubota-Leopoldt p-adic L-function which interpolates the generalized Bernoulli numbers and plays an important role in the Iwasawa theory for cyclotomic fields. Later, Snyder generalized the construction slightly and deduced a p-adic version of the reciprocity formula (4)[7]. Further Kudo constructed a p-adic interpolating function for the sums (5) and deduced many properties[8,9]. For the p-adic function constructed by Kudo, the author studied the explicit value of the radius of convergence [10]. Besides, by generalizing the sums (5) by means of Dirichlet characters, the author constructed a p-adic interpolating function for the sums and deduced the value of the radius of convergence[11].

The purpose of this paper is to extend the study of the paper[11]. We generalize the sums (6) by Dirichlet characters, construct a p-adic interpolating function and deduce the value of the radius of convergence.

Throughout the paper, we denote by $\mathbf{Q}$, $\mathbf{Z}$ and $\mathbf{N}$, the rational number field, the ring of integers of $\mathbf{Q}$ and the set of positive integers, respectively as usual, and denote the set of non-negative integers by $\bar{\mathbf{N}}$.

2 Definition of Dedekind sums attached to Dirichlet characters

As in the introduction, let B_n and $B_n(X)$ be the nth Bernoulli numbers and polynomial, respectively, defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!} \quad \text{and} \quad \frac{te^{tX}}{e^t-1} = \sum_{n=0}^{\infty} B_n(X)\frac{t^n}{n!}$$

and define $\tilde{B}_n(x) = B_n(\{x\})$.

For any primitive Dirichlet character χ, we denote by f_χ the conductor of χ and denote by I_χ the ring of rational numbers of which the denominators are relatively prime to f_χ. For any $x \in I_\chi$ we can define the value $\chi(x)$ by multiplicativity. We define the twisted Bernoulli function $\tilde{B}_{n,\chi}(x)$ attached to χ by

$$\sum_{\rho=0}^{f_\chi-1}\frac{\chi(\{x\}+\rho)te^{(\{x\}+\rho)t}}{e^{f_\chi t}-1} = \sum_{n=0}^{\infty}\tilde{B}_{n,\chi}(x)\frac{t^n}{n!}$$

or equivalently

$$\tilde{B}_{n,\chi}(x) = f_\chi^{n-1} \sum_{\rho \bmod f_\chi} \chi(x+\rho)\tilde{B}_n\left(\frac{x+\rho}{f_\chi}\right).$$

Let χ and ψ be primitive Dirichlet characters, $m,n \in \bar{\mathbf{N}}$, $h,k \in \mathbf{N}$ and $\alpha,\beta \in I_\chi \cap I_\psi$. We define the generalized Dedekind sums attached to χ and ψ by

$$S_{(m,\chi),(n,\psi)}\begin{pmatrix} h & k \\ \alpha & \beta \end{pmatrix} = \sum_{\lambda \bmod k} \tilde{B}_{m,\chi}\left(\frac{\lambda+\beta}{k}\right)\tilde{B}_{n,\psi}\left(\frac{h(\lambda+\beta)}{k} - \alpha\right). \tag{7}$$

3 Expression by Euler numbers

For a parameter u, we define the n th modified Euler numbers $E_n(u)$ for $n \in \mathbf{Z}$ with $n \geq -1$ by[9)]

$$\frac{u}{e^t - u} = \frac{E_{-1}(u)}{t} + \sum_{n=0}^{\infty} E_n(u)\frac{t^n}{n!}.$$

Note that $E_{-1}(u) \neq 0$ only if $u = 1$. We put $\tilde{n} = \max\{n, 1\}$ for $n \in \bar{\mathbf{N}}$. Then we have $\tilde{n}E_{n-1}(1) = B_n$ for $n \in \bar{\mathbf{N}}$. It is known that for any $\lambda \in \mathbf{Z}, c, k, n \in \mathbf{N}$ and for any kth root of unity ξ, we have[12)]

$$k^n \tilde{B}_n\left(\frac{\lambda}{k}\right) = \tilde{n}\sum_{\zeta^k=1} E_{n-1}(\zeta)\zeta^\lambda, \tag{8}$$

$$\tilde{n}E_{n-1}(\xi) = k^{n-1}\sum_{j \bmod k} \tilde{B}_n\left(\frac{j}{k}\right)\xi^{-j} \tag{9}$$

and

$$\sum_{\eta^c=1} E_n(u\eta) = c^{n+1}E_n(u^c). \tag{10}$$

For a primitive Dirichlet character χ, we define the numbers $E_{n,\chi}(u)$ (a modification of the generalized Euler number[13)]) by

$$\sum_{\rho=0}^{f_\chi-1} \frac{\chi(\rho)u^{f_\chi-\rho}e^{\rho t}}{e^{f_\chi t} - u^{f_\chi}} = \frac{E_{-1,\chi}(u)}{t} + \sum_{n=0}^{\infty} E_{n,\chi}(u)\frac{t^n}{n!}.$$

Note that $E_{-1,\chi}(u) \neq 0$ only if u is a primitive f_χth root of unity. Note also that $\tilde{n}E_{n-1,\chi}(1) = B_{n,\chi}$ for $n \in \bar{\mathbf{N}}$. Let ζ_χ be an arbitrarily chosen primitive f_χth root of unity and put $\tau(\chi,\zeta_\chi) = \sum_{\rho=0}^{f_\chi-1}\chi(\rho){\zeta_\chi}^\rho$, the Gauss sum attached to χ and ζ_χ. Then

$$\sum_{\rho=0}^{f_\chi-1} \frac{\chi(\rho)u^{f_\chi-\rho}e^{it}}{e^{f_\chi t} - u^{f_\chi}} = \frac{\tau(\chi,\zeta_\chi)}{f_\chi}\sum_{\rho=0}^{f_\chi-1}\frac{\chi^{-1}(\rho){\zeta_\chi}^\rho u}{e^t - {\zeta_\chi}^\rho u},$$

which implies,

$$E_{n,\chi}(u) = \frac{\tau(\chi,\zeta_\chi)}{f_\chi}\sum_{\rho \bmod f_\chi}\chi^{-1}(\rho)E_n\big({\zeta_\chi}^\rho u\big). \tag{11}$$

Hence if $\gcd\{k, f_\chi\} = 1$, as generalizations of (8), (9) and (10), we deduce that

$$\chi(k)k^n\tilde{B}_{n,\chi}\left(\frac{\lambda}{k}\right) = \tilde{n}\sum_{\zeta^k=1}E_{n-1,\chi}(\zeta)\zeta^\lambda, \tag{12}$$

$$\tilde{n}E_{n-1,\chi}(\xi) = \chi(k)k^{n-1}\sum_{j \bmod k}\tilde{B}_{n,\chi}\left(\frac{j}{k}\right)\xi^{-j} \tag{13}$$

and

$$\sum_{\eta^c=1} E_{n,\chi}(u\eta) = \chi(c)c^{n+1}E_{n,\chi}(u^c). \tag{14}$$

For $g \in \mathbf{N}$, let $J(g)$ denote an arbitrarily fixed complete set of representatives of the residue class group $\mathbf{Z}/g\mathbf{Z}$. For $g_1, \cdots g_m$, we put $J(g_1, \cdots, g_m) = J(g_1) \times \cdots \times J(g_m)$. For $c \in \mathbf{N}$ with $c > 1$, we denote by $\mathcal{V}_c$ the set of non-trivial cth roots of unity. As for the expression of the sums (7) by the Euler numbers, we have the following.

Proposition 3.1. *Let χ and ψ be primitive Dirichlet characters and let $h, k \in \mathbf{N}$. Let $\alpha, \beta \in I_\chi \cap I_\psi$ and express $\alpha = a/d$ and $\beta = b/d$ with $d \in \mathbf{N}, a, b \in \mathbf{Z}$. We suppose that* $\gcd\{h,k\} = \gcd\{kd, f_\chi f_\psi\} = 1$. *Let ζ_{kd} denote an arbitrarily chosen primitive kdth root of unity and put $\zeta_k = \zeta_{kd}^d$ and $\zeta_d = \zeta_{kd}^k$. Then we have*

$$(\chi\psi)(kd)(kd)^{m+n}S_{(m,\chi),(n,\psi)}\begin{pmatrix} h & k \\ \alpha & \beta \end{pmatrix} = \tilde{m}\tilde{n}\frac{\tau(\psi,\zeta_\psi)}{f_\psi} \times \sum_{(i,j_1,j_2,\rho)\in J(k,d,d,f_\psi)} \psi^{-1}(\rho)E_{m-1,\chi}(\zeta_{kd}^{-hi+kj_1})E_{n-1}(\zeta_\psi^\rho\zeta_{kd}^{i+kj_2})\zeta_{kd}^{bj_1-ai+j_2(hb-ka)}. \tag{15}$$

Further if $c \in \mathbf{N}$ with $c \equiv 1 \pmod{f_\psi kd}$ and $c > 1$, then

$$(c^n-1)(\chi\psi)(kd)(kd)^{m+n}S_{(m,\chi),(n,\psi)}\begin{pmatrix} h & k \\ \alpha & \beta \end{pmatrix} = \tilde{m}\tilde{n}\frac{\tau(\psi,\zeta_\psi)}{f_\psi} \times \sum_{(i,j_1,j_2,\rho)\in J(k,d,d,f_\psi)}\sum_{\eta\in\mathcal{V}_c} \psi^{-1}(\rho)E_{m-1,\chi}(\zeta_{kd}^{-hi+kj_1})E_{n-1}(\zeta_\psi^\rho\zeta_{kd}^{i+kj_2}\eta)\zeta_{kd}^{bj_1-ai+j_2(hb-ka)}. \tag{16}$$

Proof. We see from (12) that

$$\begin{aligned}\chi(kd)(kd)^m\tilde{B}_{m,\chi}\left(\frac{\lambda+\beta}{k}\right) &= \chi(kd)(kd)^m\tilde{B}_{m,\chi}\left(\frac{\lambda d+b}{kd}\right) \\ &= \tilde{m}\sum_{(i_1,j_1)\in J(k,d)} E_{m-1,\chi}(\zeta_{kd}^{-hi_1+kj_1})\zeta_{kd}^{(-hi_1+kj_1)(\lambda d+k)}\end{aligned}$$

and

$$\begin{aligned}\psi(kd)(kd)^n\tilde{B}_{m,\psi}\left(\frac{h(\lambda+\beta)}{k}-\alpha\right) &= \psi(kd)(kd)^n\tilde{B}_{n,\psi}\left(\frac{h\lambda d+hb-ka}{kd}\right) \\ &= \tilde{n}\sum_{(i_2,j_2)\in J(k,d)} E_{n-1,\psi}(\zeta_{kd}^{i_2+kj_2})\zeta_{kd}^{(i_2+kj_2)(h(\lambda d+b)-ka)}.\end{aligned}$$

Note that for $i_1, i_2 \in J(k)$ we have $\sum_{\lambda\in J(k)}\zeta_{kd}^{h(-i_1+i_2)(\lambda d+b)} = k$ or 0 according as $i_1 = i_2$ or $i_1 \neq i_2$. Hence by (7) and (11), we obtain (15). In addition, applying (14), we also obtain (16).

4 p-adic interpolation

Let p be a prime number. If $p \geq 3$, we put $e_o = p-1$ and $q = p$. If $p = 2$, we put $e_0 = 2$ and $q = 4$. In this section we construct a p-adic interpolating function for the sums (7).

As usual, we denote by $\mathbf{Q}_p$, $\mathbf{Z}_p$ and $\mathbf{C}_p$ the rational p-adic number field, the ring of integers of $\mathbf{Q}_p$ and the completion of the algebraic closure of $\mathbf{Q}_p$, respectively. Let $|\ \ |_p$ denote the p-adic valuation of $\mathbf{C}_p$ normalized by $|p|_p = 1/p$. For any $u \in \mathbf{C}_p{}^\times$ with $|1-u^p|_p \geq 1$, the Koblitz measure $\mathcal{M}_u$ on $\mathbf{Z}_p$ is defined by

$$\mathcal{M}_u(\nu + p^n\mathbf{Z}_p) = \frac{u^{p^n-\nu}}{1-u^{p^n}}$$

for any $n, \nu \in \bar{\mathbf{N}}$ with $0 \le \nu \le p^n - 1$ and we have

$$\int_{\mathbf{Z}_p} x^n \, d\mathcal{M}_u(x) = E_n(u) \quad \text{and} \quad \int_{\mathbf{Z}_p^\times} x^n \, d\mathcal{M}_u(x) = E_n(u) - p^n E_n(u^p).$$

Let ω_p denote the Teichmüller character for p and put $\langle x \rangle = x/\omega_p(x)$ for $x \in \mathbf{Z}_p^\times$. We put

$$G_p(s,u) = \int_{\mathbf{Z}_p^\times} \langle x \rangle^s d\mathcal{M}_u(x) \ \text{ for } \ s \in \mathbf{Z}_p, \tag{17}$$

which is the p-adic Γ-transform for the measure $\mathcal{M}_u$ and satisfies an interpolating property such as

$$G_p(n,u) = E_{n,\omega_p^{-n}}(u) - p^n E_{n,\omega_p^{-n}}(u^p) \tag{18}$$

for $n \in \bar{\mathbf{N}}$.

As in Proposition 3.1, let χ and ψ be primitive Dirichlet characters and let $h, k \in \mathbf{N}$. Let $\alpha, \beta \in I_\chi \cap I_\psi$ and express $\alpha = a/d$ and $\beta = b/d$ with $d \in \mathbf{N}, a, b \in \mathbf{Z}$. We suppose that $\gcd\{h,k\} = \gcd\{kd, f_\chi f_\psi\} = \gcd\{p, kdf_\chi f_\psi\} = 1$. In addition, we choose and fix integers $c, p' \in \mathbf{N}$ with $c \equiv 1 \pmod{qkdf_\psi}$, $c > 1$ and $pp' \equiv 1 \pmod{kdf_\psi}$. For each $m \in \bar{\mathbf{N}}$, we set

$$T^c_{p,m}\left(s, \chi, \psi : \begin{pmatrix} h & k \\ \alpha & \beta \end{pmatrix}\right) = \tilde{m}k \frac{\tau(\psi, \zeta_\psi)}{f_\psi} \times \sum_{(i,j_1,j_2) \in J(k,d,d)} \sum_{\eta \in \mathcal{V}_c} E_{m-1,\chi}(\zeta_{kd}^{-h_i + kj_1}) \zeta_d^{bj_1 - ai + j_2(hb-ka)} G_p(s, \zeta_{kd}^{i+kj_2}\eta). \tag{19}$$

Then by Proposition 3.1 and (18), we deduce that

$$T^c_{p,m}\left(n-1, \chi, \psi : \begin{pmatrix} h & k \\ \alpha & \beta \end{pmatrix}\right) = \tilde{m}\tilde{n}k(c^n - 1)(\chi\psi)(kd)(kd)^{m+n} \times \left(S_{(m,\chi),(n,\psi)} \begin{pmatrix} h & k \\ \alpha & \beta \end{pmatrix} - \psi^{-1} p^n S_{(m,\chi),(n,\psi)} \begin{pmatrix} p'h & k \\ p'\alpha & \beta \end{pmatrix} \right)$$

for any $n \in \mathbf{N}$ with $n \equiv 1 \pmod{e_0}$. Now we define

$$S_{p,m}\left(s, \chi, \psi : \begin{pmatrix} h & k \\ \alpha & \beta \end{pmatrix}\right) = \frac{1}{\tilde{m}k(c^s - 1)(\chi\psi)(kd)(kd)^m d < d >^s} T^c_{p,m}\left(s, \chi, \psi : \begin{pmatrix} h & k \\ \alpha & \beta \end{pmatrix}\right), \tag{20}$$

which is independent of the choice of c. Then we obtain the following.

THEOREM 4.1. *We have the interpolating property such as*

$$S_{p,m}\left(n-1, \chi, \psi : \begin{pmatrix} h & k \\ \alpha & \beta \end{pmatrix}\right) = \tilde{n}k^n \left(S_{(m,\chi),(n,\psi)} \begin{pmatrix} h & k \\ \alpha & \beta \end{pmatrix} - \psi^{-1}(p) p^n S_{(m,\chi),(n,\psi)} \begin{pmatrix} p'h & k \\ p'\alpha & \beta \end{pmatrix} \right)$$

for any $n \in \mathbf{N}$ *with* $n \equiv 1 \pmod{e_0}$.

5 Radius of convergence

By (17), the function $G_p(s,u)$ is expanded at any $s_0 \in \mathbf{Z}_p$ as

$$G_p(s,u) = \sum_{n=0}^{\infty} c_{n,u,s_0}(s - s_0)^n \tag{21}$$

with

$$c_{n,u,s_0}=\frac{1}{n!}\int_{\mathbf{Z}_p^\times}(\log_p <x>)^n <x>^{s_0}\,d\mathcal{M}_u(x).$$

Hence we can enlarge the domain of definition of the function $G_p(s,u)$ from $\mathbf{Z}_p$ to the set of $s\in\mathbf{C}_p$ for which the right-hand side of (21) converges. For the same reason, the domain of definition of the function $S_{p,m}\left(s,\chi,\psi:\begin{pmatrix}h&k\\ \alpha&\beta\end{pmatrix}\right)$ can be enlarged. Let $r_m(\chi,\psi,h,k,\alpha,\beta:s_0)$ denote the radius of convergence of the expansion of $S_{p,m}\left(s,\chi,\psi:\begin{pmatrix}h&k\\ \alpha&\beta\end{pmatrix}\right)$ at $s_0\in\mathbf{Z}_p$. In order to study the value, we first recall the main result of Section 3 of the paper[10).

Let $\mathcal{I}$ be a finite set and consider functions

$$U:\mathcal{I}\to\mathbf{C}_p^\times \ \text{ and } \ \mathcal{A}:\mathcal{I}\to\mathbf{C}_p^\times.$$

For each $i\in\mathcal{I}$, put $U(i)=u_i$ and $\mathcal{A}(i)=\alpha_i$ and suppose that $|1-u_i^p|_p\geq 1$ for all $i\in\mathcal{I}$. We put

$$G_p(s:U,\mathcal{A})=\sum_{i\in I}\alpha_i G_p(s,u_i)$$

and denote by $r(U,\mathcal{A}:s_0)$ the radius of convergence of $G_p(s:U,\mathcal{A})$ at $s_0\in\mathbf{Z}_p$. Let $\mathcal{I}_+=\{i\in\mathcal{I}||u_i|_p<1\}$, $\mathcal{I}_-=\{i\in\mathcal{I}||u_i|_p>1\}$ and $\mathcal{I}_0=\{i\in\mathcal{I}||u_i|_p=1\}$. For each $n\in\mathbf{N}$, we put

$$\mathcal{N}_n=\{\mu\in\mathbf{N}|\gcd\{\mu,p\}=1,|<\mu>-1|_p=|q|_p|p|_p^{n-1}\}.$$

If there exists an integer $n\in\mathbf{N}$ such that either

$$\sum_{i\in\mathcal{I}_+}\alpha_i u_i^\mu-\sum_{i\in\mathcal{I}_-}\alpha_i u_i^{-\mu}\neq 0 \ \text{ or } \ \sum_{i\in\mathcal{I}_0}\alpha_i(u_i^\mu-u_i^{-|\mu})\neq 0$$

holds for some $\mu\in\mathcal{N}_n$, we denote the minimum of such n by $n(U,\mathcal{A})$. Otherwise we put $n(U,\mathcal{A})=\infty$. Then we have[10)

$$r(U,\mathcal{A}:s_0)=\begin{cases}|p|_p^{\frac{1}{p-1}-n(U,\mathcal{A})+1}|q|_p^{-1} & \text{if } n(U,\mathcal{A})\neq\infty.\\ \infty & \text{if } n(U,\mathcal{A})=\infty.\end{cases}\tag{22}$$

Now we put $r\left(S_{p,m}\left(s;\chi,\psi\begin{pmatrix}h&k\\ \alpha&\beta\end{pmatrix}\right):s_0\right)=r_m(\chi,\psi,h,k,\alpha,\beta:s_0)$ for $s_0\in\mathbf{Z}_p$. Further, we introduce following subsets of $\mathbf{Q}$:

$$\mathcal{B}_{m,\chi}=\{x\in\mathbf{Q}|\tilde{B}_{m,\chi}(x)=0\} \text{ and } \mathcal{C}_{m,\chi}(\varepsilon,A)=\{x\in\mathbf{Q}|\tilde{B}_{m,\chi}(A+x)-\varepsilon\tilde{B}_{m,\chi}(A-x)=0\} \text{ for } A\in\mathbf{Q}.$$

The main result is the following.

THEOREM 5.1. *Let $h'\in\mathbf{Z}$ be an arbitrary integer such that $hh'\equiv 1 \pmod k$.*
(1) *If $2(h\beta-k\alpha)\notin\mathbf{Z}$, then*

$$r_m(\chi,\psi,h,k,\alpha,\beta:s_0)=\begin{cases}|p|_p^{\frac{1}{p-1}}|q|_p^{-1} & \text{if } \dfrac{\lambda+\beta}{k}\notin\mathcal{B}_{n,\chi} \text{ for some } \lambda\in\mathbf{Z}.\\ \infty & \text{otherwise.}\end{cases}$$

(2) If $2(h\beta-k\alpha)\in\mathbf{Z}$, then

$$r_m(\chi,\psi,h,k,\alpha,\beta:s_0)=\begin{cases}|p|_p^{\frac{1}{p-1}}|q|_p^{-1} & \text{if } \dfrac{-h'(h\beta-k\alpha)+\lambda}{k}\notin\mathcal{C}_{n,\chi}\left(\psi(-1),\dfrac{\beta-h'(h\beta-k\alpha)}{k}\right) \text{ for some } \lambda\in\mathbf{Z}.\\ \infty & \text{otherwise.}\end{cases}$$

Proof. By (19) and (20), it is sufficient to prove the assertion for $T^c_{p,m}\left(s,\chi,\psi:\begin{pmatrix}h & k\\ \alpha & \beta\end{pmatrix}\right)$ instead of $S_{p,m}\left(s,\chi,\psi:\begin{pmatrix}h & k\\ \alpha & \beta\end{pmatrix}\right)$. We put $\mathcal{I}=J(f_\psi)\times J(k)\times J(d)\times\mathcal{V}_c$. For each $(\rho,i,j_2,\eta)\in\mathcal{I}$, we put

$$u(\rho,i,j_2,\eta)=\zeta_\psi^\rho\zeta_{kd}^{i+kj_2}\eta \text{ and } \alpha(\rho,i,j_2,\eta)=\sum_{j_1\in J(d)}\psi^{-1}(\rho)E_{m-1,\chi}(\zeta_{kd}^{-hi+kj_1})\zeta_d^{bj_1-ai+j_2(hb-ka)}.$$

Then by (19), we see that

$$T^c_{p,m}\left(s;\chi,\psi\begin{pmatrix}h & k\\ \alpha & \beta\end{pmatrix}\right)=\sum_{(\rho,i,j_2,\eta)\in I}\alpha(\rho,i,j_2,\eta)G_p(s,u(\rho,i,j_2,\eta)).$$

Note that $|u(\rho,i,j_2,\eta)|_p=1$ for all $(\rho,i,j_2,\eta)\in\mathcal{I}$. For $\mu\in\mathbf{N}$, we put

$$\begin{aligned}\mathcal{F}(\mu) &= \sum_{(\rho,i,j_2,\eta)\in\mathcal{I}}\alpha(\rho,i,j_2,\eta)(u(\rho,i,j_2,\eta)^\mu-u(\rho,i,j_2,\eta)^{-\mu})\\ &=\sum_{(\rho,i,j_2,\eta)\in\mathcal{I}}\sum_{j_1\in J(d)}\psi^{-1}(\rho)E_{m-1,\chi}(\zeta_{kd}^{-hi+kd})\zeta_d^{bj_1-ai+j_2(hb-ka)}\\ &\qquad\times((\zeta_\psi^\rho\zeta_{kd}^{i+kj_2}\eta)^\mu-(\zeta_\psi^\rho\zeta_{kd}^{i+kj_2}\eta)^{-\mu}).\end{aligned}$$

Note that $\sum_{\rho\in J(f_\psi)}\psi^{-1}(\rho)\zeta_\psi^{\pm\rho\mu}=\psi(\pm\mu)\tau(\psi^{-1},\zeta_\psi)$. For any $x\in\mathbf{Q}$, put $\Phi(x)=1$ or $\Phi(x)=0$ according as $x\in\mathbf{Z}$ or $x\notin\mathbf{Z}$. Then

$$\sum_{\eta\in\mathcal{J}_c}\eta^{\pm\mu}=\Phi\left(\frac{\mu}{c}\right)c-1 \text{ and } \sum_{j_2\in J(d)}\zeta_d^{j_2(hb-ka)}\zeta_{kd}^{\pm kj_2\mu}=\Phi\left(\frac{hb-ka\pm\mu}{d}\right)d.$$

Further for any $r_1,r_2\in\mathbf{C}_p$, let us write $r_1\sim r_2$ if $r_1=r_2r$ for some $r\in\mathbf{C}_p^\times$. Then we can express

$$\mathcal{F}(\mu)\sim\psi(\mu)\sum_{(i,j_1)\in J(k)\times J(d)}E_{m-1,\chi}(\zeta_{kd}^{-hi+kj_1})\times\zeta_d^{bj_1-ai}\left(\zeta_{kd}^{i\mu}\Phi\left(\frac{hb-ka+\mu}{d}\right)-\psi(-1)\zeta_{kd}^{-i\mu}\Phi\left(\frac{hb-ka-\mu}{d}\right)\right).$$

Applying (13) we also deduce that

$$\mathcal{F}(\mu)\sim\psi(\mu)\sum_{(i,j_1)\in J(k)\times J(d)}\sum_{\tau\in J(kd)}\tilde{B}_{m,\chi}\left(\frac{\tau}{kd}\right)\zeta_{kd}^{(hi-kj_1)\tau}\times\zeta_d^{bj_1-ai}\left(\zeta_{kd}^{i\mu}\Phi\left(\frac{hb-ka+\mu}{d}\right)-\psi(-1)\zeta_{kd}^{-i\mu}\Phi\left(\frac{hb-ka-\mu}{d}\right)\right).$$

Note that $\sum_{j_1\in J(d)}\zeta_{kd}^{-kj_1\tau}\zeta_d^{bj_1}=\sum_{j_1\in J(d)}\zeta_d^{(b-\tau)j_1}=\Phi((b-d)/\tau)d$. Hence

$$\mathcal{F}(\mu)\sim\psi(\mu)\sum_{i\in J(k)}\sum_{\tau\in J(k)}\tilde{B}_{m,\chi}\left(\frac{b+\tau d}{kd}\right)\times\left(\zeta_{kd}^{(hb-ka+\mu+h\tau d)i}\Phi\left(\frac{hb-ka+\mu}{d}\right)-\psi(-1)\zeta_{kd}^{(hb-ka-\mu+h\tau d)i}\Phi\left(\frac{hb-ka-\mu}{d}\right)\right).$$

If $hb - ka \pm \mu \not\equiv 0 \pmod d$, then $\mathcal{F}(\mu) = 0$. If $hb - ka + \mu \equiv 0 \pmod d$, we can express $hb - ka + \mu = \tau_1 d$ for some $\tau_1 \in \mathbf{Z}$. In addition, if $2(hb - ka) \not\equiv 0 \pmod d$, then $hb - ka - \mu = 2(hb - ka) - \tau_1 d \not\equiv 0 \pmod d$ and we see that

$$\mathcal{F}(\mu) \sim \psi(\mu) \sum_{i \in J(k)} \sum_{\tau \in J(k)} \tilde{B}_{m,\chi}\left(\frac{b + \tau d}{kd}\right) \zeta_k^{(\tau_1 + h\tau)i} \sim \psi(\mu) \tilde{B}_{m,\chi}\left(\frac{\beta - h'\tau_1}{k}\right).$$

If $(\lambda_1 + \beta)/k \notin \mathcal{B}_{m,\chi}$ for some $\lambda_1 \in \mathbf{Z}$, then by the assumption $\gcd\{f_\psi p, kd\} = 1$ there is an integer $\mu_1 \in \mathcal{N}_1$ satisfying $\mu_1 \equiv -h\lambda_1 d - hk + ka \pmod{kd}$ and $\gcd\{\mu_1, f_\psi\} = 1$. For such μ_1, we have

$$\mathcal{F}(\mu_1) \sim \tilde{B}_{m,\chi}\left(\frac{\lambda_1 + \beta}{k}\right).$$

This implies $\mathcal{F}(\mu_1) \neq 0$ and by (22), we conclude that $r_m(\chi, \psi, h, k, \alpha, \beta : s_0) = |p|_p^{\frac{1}{p-1}} |q|_p^{-1}$. On the other hand if $(\lambda + \beta)/k \in \mathcal{B}_{m,\chi}$ for all $\lambda \in \mathbf{Z}$, considering the case of $hb - ka - \mu \equiv 0 \pmod d$ in the same way, we see that $\mathcal{F}(\mu) = 0$ for all $\mu \in \mathbf{N}$. Hence we conclude that $r_m(\chi, \psi, h, k, \alpha, \beta : s_0) = \infty$.

Finally let us consider the case of $2(hb - ka) \equiv 0 \pmod d$. In this case if $hb - ka + \mu \equiv 0 \pmod d$, we can express $hb - ka + \mu = \tau_1 d$ and $hb - ka - \mu = 2(hb - ka) - \tau_1 d$ for some $\tau_1 \in \mathbf{Z}$ and

$$\begin{aligned} \mathcal{F}(\mu) &\sim \psi(\mu)\left(\tilde{B}_{m,\chi}\left(\frac{\beta - h'\tau_1}{k}\right) - \psi(-1)\tilde{B}_{m,\chi}\left(\frac{\beta - 2h'(h\beta - k\alpha) + h'\tau_1}{k}\right)\right) \\ &\sim \psi(\mu)\Biggl(\tilde{B}_{m,\chi}\left(\frac{\beta - h'(h\beta - k\alpha)}{k} + \frac{-h'(h\beta - k\alpha) + h'\tau_1}{k}\right) \\ &\qquad -\psi(-1)\tilde{B}_{m,\chi}\left(\frac{\beta - h'(h\beta - k\alpha)}{k} - \frac{-h'(h\beta - k\alpha) + h'\tau_1}{k}\right)\Biggr). \end{aligned}$$

Hence in the similar way as in the case of $2(hb - ka) \not\equiv 0 \pmod d$, we obtain our assertion. This completes the proof.

If χ is trivial, making use of some fundamental properties of Bernoulli numbers and polynomials, we can deduce more explicit results as the following.

Corollary 5.2. *If χ is trivial. we have*

$$r_m(\chi, \psi, h, k, \alpha, \beta : s_0) = |p|_p^{\frac{1}{p-1}} |q|_p^{-1}$$

except for the following cases :

Case 1: $2(h\beta - k\alpha) \notin \mathbf{Z}$, $m \equiv 1 \pmod 2$, $k = 1$, $\beta \in \frac{1}{2} + \mathbf{Z}$.
Case 2: $2(h\beta - k\alpha) \notin \mathbf{Z}$, $m \equiv 1 \pmod 2$ with $m \geq 3$, $k \leq 2$, $\beta \in \mathbf{Z}$.
Case 3: $2(h\beta - k\alpha) \in \mathbf{Z}$, $\psi(-1) = 1$, $k = 1$.
Case 4: $2(h\beta - k\alpha) \in \mathbf{Z}$, $\psi(-1) = 1$, $k = 2$, $m \geq 2$, $h\beta - k\alpha \in \mathbf{Z}$.
Case 5: $2(h\beta - k\alpha) \in \mathbf{Z}$, $\psi(-1) = 1$, $k = 2$, $m \geq 2$, $h\beta - k\alpha, \beta \in \frac{1}{2} + \mathbf{Z}$.
Case 6: $2(h\beta - k\alpha) \in \mathbf{Z}$, $\psi(-1) = 1$, $k = 2$, $m \equiv 1 \pmod 2$ with $m \geq 3$, $\beta \in \mathbf{Z}$.
Case 7: $2(h\beta - k\alpha) \in \mathbf{Z}$, $\psi(-1) = 1$, $k \geq 3$, $m \equiv 0 \pmod 2$, $2\alpha, 2\beta \in \mathbf{Z}$.
Case 8: $2(h\beta - k\alpha) \in \mathbf{Z}$, $\psi(-1) = -1$, $k = 1$, $m = 1$, $\beta \in \frac{1}{2} + \mathbf{Z}$.
Case 9: $2(h\beta - k\alpha) \in \mathbf{Z}$, $\psi(-1) = -1$, $k = 1$, $m \equiv 1 \pmod 2$ with $m \geq 3$, $2\beta \in \mathbf{Z}$.
Case 10: $2(h\beta - k\alpha) \in \mathbf{Z}$, $\psi(-1) = -1$, $k = 2$, $m = 1$, $2\alpha \in \mathbf{Z}$, $\beta \in \frac{1}{2} + \mathbf{Z}$.
Case 11: $2(h\beta - k\alpha) \in \mathbf{Z}$, $\psi(-1) = -1$, $k = 2$, $m \equiv 1 \pmod 2$ with $m \geq 3$, $2\alpha, 2\beta \in \mathbf{Z}$.
Case 12: $2(h\beta - k\alpha) \in \mathbf{Z}$, $\psi(-1) = -1$, $k = 2$, $m \equiv 1 \pmod 2$ with $m \geq 3$, $2\alpha \in \frac{1}{2} + \mathbf{Z}, \beta \in \mathbf{Z}$.

Case 13: $2(h\beta - k\alpha) \in \mathbf{Z}$, $\psi(-1) = -1$, $k \geq 3$, $m = 1$, $2\alpha, 2\beta \in \mathbf{Z}$.
Case 14: $2(h\beta - k\alpha) \in \mathbf{Z}$, $\psi(-1) = -1$, $k \geq 3$, $m \equiv 1 \pmod 2$ with $m \geq 3$, $2\alpha, \beta \in \mathbf{Z}$.

In these exceptional cases, we have

$$S_{p,m}\left(s;\chi,\psi\begin{pmatrix} h & k \\ \alpha & \beta \end{pmatrix}\right) = 0 \quad (\textit{identically}).$$

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