Connectivities of Networks and Subnetworks

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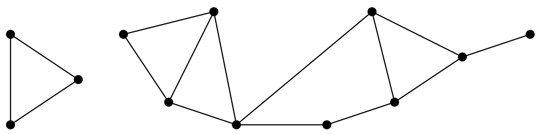
1 Purpose

To provide a relation between connectivities of a network and connectivities of some subnetworks.

2 Network

$G = (V, E)$: a network (V: the node set, E: the edege set)
Network theory has application in many disciplines including statistical physics, particle physics, computer science, biology, economics, operations research, and sociology.

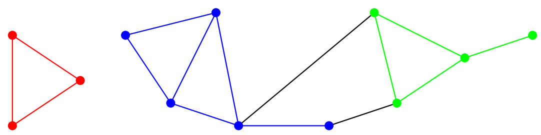


3 Connectivity of Network

$G = (V, E)$: a network
k: a natural number ≥ 2
The k-way expansion constant $h_k(G)$ of G is defined as

$$h_k(G) := \min \left\{ \max_{i=1,2,\ldots,k} \frac{|\partial V^i|}{|V^i|} : V = \bigsqcup_{i=1}^{k} V^i, V^i \neq \emptyset \right\},$$

where ∂F is the set of the edges connecting F and $V - F$.

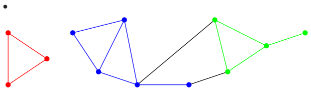


$G^1 = (V^1, E^1)$ $G^2 = (V^2, E^2)$ $G^3 = (V^3, E^3)$

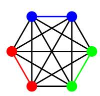
The k-way expansion constant $h_k(G)$ is a strength of connectivity of G with respect to partitions into k subnetworks.

4 Examples

Example 1 (Unconnected network)**.**
The number of the connected component of G is k if and only if $h_k(G) = 0$ and $h_{k+1}(G) > 0$.

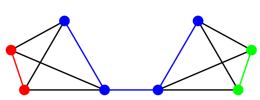


Example 2 (Complete graph)**.**
For $n, k \in \mathbb{N}$, $h_k(K^{kn}) = (k-1)n$.
Connectivity of complete graphs are very strong.



K^6

Example 3.
Let $G_{n,m} := (V_{K^n} \cup V_{K^m}, E_{K^n} \cup E_{K^m} \cup \{vw\})$ for $n, m \in \mathbb{N}$, where $v \in V_{K^n}$ and $w \in V_{K^m}$. Then $h(G_{n,m}) = 1/\min\{n, m\}$.
For $n \in \mathbb{N}$, $h_3(G_{2n,2n}) = n$, $h_2(G_{2n,2n}) = 1/2n$.



$G_{4,4}$

5 Relation with eigenvalues of Laplacian

It is known that the k-way expansion constant $h_k(G)$ is estimated by the k-th eigenvalue of the Laplacian on the network G:

Theorem 4 (Lee-Gharan-Trevisan (2012))**.** *There is a constant $C > 0$ such that*

$$\frac{\lambda_k(G)}{2\deg(G)} \leq h_k(G) \leq Ck^2 \deg(G)\sqrt{\lambda_k(G)} \tag{1}$$

for every connected networks G and every $k \in \{2, \ldots, |V|\}$, where $\deg(G)$ is the maximum number of edges incident to a node among its nodes.

6 Result

Lemma 5. *For any k-partition $\{G^i = (V^i, E^i)\}_{i=1}^k$ of G*

$$h_{k+1}(G) \geq \min_{i=1,2,\ldots,k} h_2(G^i).$$

Theorem 6. *If $h_{k+1}(G)/3^{k+1} > h_k(G)$ for some k, then there exists a k-partition $\{G^i = (V^i, E^i)\}_{i=1}^k$ of G satisfying*

$$\frac{h_{k+1}(G)}{3^{k+1}} \leq \min_{i=1,2,\ldots,k} h_2(G^i),$$

$$\max_{i=1,2,\ldots,k} \frac{|\partial V^i|}{|V^i|} \leq 3^k h_k(G).$$

This theorem says that if $h_{k+1}(G)$ is sufficiently large and $h_k(G)$ sufficiently small, then there is a k-partition such that every subgraphs in it have strong connectivity and connectivity between them is weak.

Similar theorem for a manifold with respect to k-th eigenvalues of the Laplacian on it was given by Funano-Shioya.

7 Coarse non-embeddability

We can define a natural distance to networks giving distance 1 to each edge.

A sequence of metric spaces $\{(X_n, d_{X_n})\}_{n=1}^\infty$ is said to be *coarsely embeddable* into a metric space (Y, d_Y) if there exist two non-decreasing functions $\rho_1, \rho_2 : [0, +\infty) \to \mathbb{R}$ and maps $\{f_n : X_n \to Y\}_{n=1}^\infty$ such that

1. $\rho_1(d_{X_n}(x, y)) \leq d_Y(f_n(x), f_n(y)) \leq \rho_2(d_{X_n}(x, y))$ for all $x, y \in X_n$ and n;
2. $\lim_{r\to\infty} \rho_1(r) = +\infty$.

Corollary 7. *If a sequence of networks $\{G_n = (V_n, E_n)\}_{n=1}^\infty$ satisfies (i) $|V_n| \to \infty$ as $n \to \infty$; (ii) $\sup_{n\in\mathbb{N}} \deg(G_n) < \infty$; (iii) $\inf_{n\in\mathbb{N}} h_{k+1}(G_n) > 0$ for some $k \in \mathbb{N}$, then it is not coarsely embeddable into any Hilbert space.*

This means if $|V|$ is large, $\deg(G)$ is small and $h_{k+1}(G)$ is not small, then the metric structure of G is very different to the metric structure of Euclid space.