The existence of a global fixed point of an isometric action on a strictly convex Banach space

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A topological group is said to have Property (FH) if every isometric action of the group on an infinite dimensional real Hilbert space has a global fixed point. For a locally compact second countable group, Property (FH) is equivalent to Kazhdan's property (T), which plays an important role in many subjects (see [BHV08]). Note that a locally compact second countable group which has Property (FH) is compactly generated. A topological group is said to have Property (F_B) if every isometric action of the group on a Banach space B has a global fixed point. In [BFGM07], they point out that a locally compact second countable group has Property (FH) if and only if it has Property (F_{L^p}) for $1 < p < 2 + \epsilon$, where ϵ depends on the group. G. Yu [Yu05] proves that any hyperbolic group does not have Property (F_{L^p}) for sufficiently large p depending on the group. M. Mimura [Mim] proves that for $k \geq 0$ and $n \geq 4$, $SL_n(\mathbb{Z}[X_1, \ldots, X_k])$ has property (F_{L^p}) for $1 < p < \infty$. In this talk, we discuss the existence of global fixed points of isometric actions of a finitely generated group on strictly convex real Banach spaces.

Let G be a finitely generated group and K a finite generating subset of G. Let $(B, \| \ \|)$ be a strictly convex real Banach space, that is, a real Banach space whose unit sphere has no non-trivial segment. For example, the Lebesgue spaces L^p with $1 < p < \infty$ are strictly convex. For an isometric action α on B and $1 < p < \infty$, set

$$F_{\alpha,p}(v) := \left(\sum_{g \in K} \frac{\|v - \alpha(g, v)\|^p}{|K|} \right)^{1/p}$$

and

$$|\nabla_- F_{\alpha,p}|(v) := \max \left\{ \limsup_{u \to v, u \in B} \frac{F_{\alpha,p}(v) - F_{\alpha,p}(u)}{\|v - u\|}, 0 \right\}$$

for $v \in B$. The function $F_{\alpha,p}$ vanishes at v if and only if v is a global fixed point of α. The function $|\nabla_- F_{\alpha,p}|$ can be regarded as the norm of the gradient of $F_{\alpha,p}$.

Theorem. *Let $\mathcal{L}$ be a family of strictly convex real Banach spaces which is stable under ultralimit, and $1 < p < \infty$. Then the following are equivalent:*

(i) *For any Banach space $B \in \mathcal{L}$, G has property (F_B);*

(ii) *for any Banach space $B \in \mathcal{L}$ and any isometric action α on B there exists a constant $C > 0$ such that $|\nabla_- F_{\alpha,p}|(v) \geq C$ for all $v \in B$ with $F_{\alpha,p}(v) > 0$.*

We can take the constant $C = C(\Gamma, \mathcal{L})$ independently of B and α.

The family of all Hilbert spaces is an example of a family of strictly convex real Banach spaces which is stable under ultralimit. For this family, this theorem is known, and (i) in this theorem is equivalent to that G has Property (FH). For fixed p with $1 < p < \infty$, the family of all L^p spaces is also an example (see [AK90] and [Hei80]).

For $L^p(\nu)$ with $1 < p < \infty$, we can show

$$|\nabla_- F_{\alpha,p}|(f) = \frac{2}{F_{\alpha,p}(f)^{p-1}} \left\| \sum_{g \in K} \frac{|f - \alpha(g)f|^{p-2}(f - \alpha(g)f)}{|K|} \right\|_{L^q(\nu)}$$

for $f \in L^p(\nu)$ with $F_{\alpha,p}(f) > 0$, where $q = p/(p-1)$ and for $p < 2$ we set $|h(x)|^{p-2}(h(x)) = 0$ if $h(x) = 0$.

References

[AK90] A. G. Aksoy and M. A. Khamsi, *Nonstandard Methods in Fixed point Theory*, Springer-Verlag, New York, 1990.

[BFGM07] U. Bader, A. Furman, T. Gelander, and N. Monod, *Property (T) and rigidity for actions on Banach spaces*, Acta Math. **198** (2007), no.1, 57–105.

[BHV08] B. Bekka, P. de la Harpe, A. Valette, *Kazhdan's property (T)*, New Mathematical Monographs, 11. Cambridge University Press, Cambridge, 2008.

[Hei80] S. Heinrich, *Ultraproduct in Banach space theory*, J. Reine and Ang. Mat. **313** (1980), 72–104.

[Mim] M. Mimura, *Fixed point properties and second bounded cohomology of universal lattices on Banach space*, preprint arXiv:0904.4650.

[Yu05] G. Yu, *Hyperbolic groups admit proper affine isometric actions on ℓ^p-spaces*, Geom. Funct. Anal. **15** (2005), 1144–1151.