Higher eigenvalues of the Laplacian on a graph and partitions of the graph

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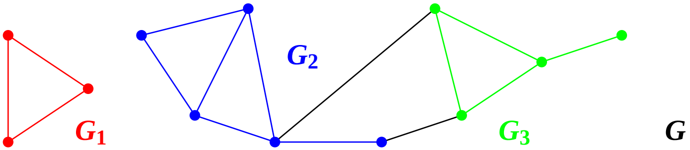
$$\lambda_l(G) \quad \longleftrightarrow \quad h_l(G) \quad \longleftrightarrow \quad \min_{i=1,2,\ldots,l} \lambda_2(G_i)$$

$\lambda_l(G)$: l-th eigenvalue of the Laplacian on a finite graph G
$h_l(G)$: the expander constant of G (a quantity of some connectivity)
$\{G_i\}_{i=1}^{l}$: a partition of G



Expander constant

$G = (V, E)$: a finite graph, i.e.,

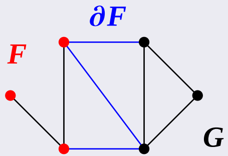
V is a finite set, which is called the vertex set,

$E \subset \{xy : x, y \in V, x \neq y\}$, where $xy = yx$, the edge set.

Definition

The *expander constant* of G is

$$h(G) = \min_{F \subset V} \left\{ \frac{|\partial F|}{|F|} : 1 \leq |F| \leq \frac{|V|}{2} \right\}$$



where ∂F is the set of the edges connecting F and $V - F$.

This represents strength of connection between two disjoint vertex sets.

Example

K^m denotes a complete graph, i.e.,

$$|V_{K^m}| = m \text{ and } E_{K^m} = \{xy : x, y \in V_{K^m}, x \neq y\}.$$

Let G_{n_1,n_2} be a graph with $V_{G_{n_1,n_2}} = V_{K^{n_1}} \cup V_{K^{n_2}}$ and $E_{G_{n_1,n_2}} = E_{K^{n_1}} \cup E_{K^{n_2}} \cup \{xy\}$ for some $x \in V_{K^{n_1}}$ and $y \in V_{K^{n_2}}$.
Then for $n, n_1, n_2 \in \mathbb{N}$

$$h(K^{2n}) = n \geq 1, \quad h(G_{n_1,n_2}) = \frac{1}{\min\{n_1, n_2\}} \leq 1.$$

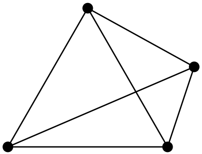


Figure: K^4

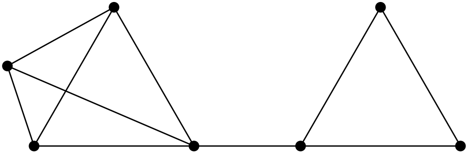


Figure: $G_{4,3}$

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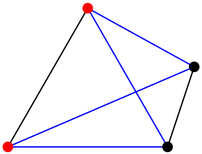

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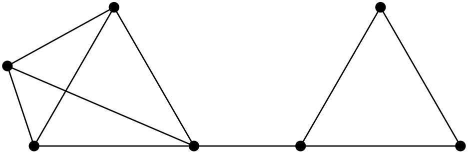

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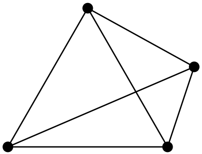

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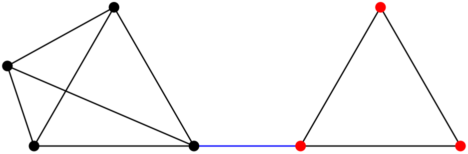


Figure: $G_{4,3}$

Eigenvalues of Laplacian

$\deg(x) := |\{y \in V : xy \in E\}|$ for $x \in V$.

Definition

The *Laplacian* on G is the $|V| \times |V|$-matrix $\Delta_G := D(G) - A(G)$,

$$D(G)_{ij} := \begin{cases} \deg(x_i) & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases} \qquad A(G)_{ij} := \begin{cases} 1 & \text{if } x_i x_j \in E \\ 0 & \text{otherwise,} \end{cases}$$

where $V = \{x_1, x_2, \ldots, x_{|V|}\}$.

$\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_{|V|}(G)$: the eigenvalues of Δ_G

- G is connected $\Leftrightarrow \lambda_2(G) > 0$.
- The number of connected components of G is l $\Leftrightarrow \lambda_l(G) = 0$ and $\lambda_{l+1}(G) > 0$.

Question

$\deg(G) := \max_{x \in V} \deg(x)$.

Theorem (Alon-Milman, Dodziuk)

$$\frac{\lambda_2(G)}{2} \leq h(G) \leq \sqrt{2\deg(G)\lambda_2(G)}. \tag{1}$$

This theorem implies that the second eigenvalue $\lambda_2(G)$ also represents some strength of connection between two disjoint subgraphs.

Question

For $l > 2$, can we relate $\lambda_l(G)$ to some connectivity of a graph and subgraphs as the inequalities (1)*?*

$$\lambda_l(G) \overset{?}{\longleftrightarrow} \text{Some connectivity of } G$$

Partition

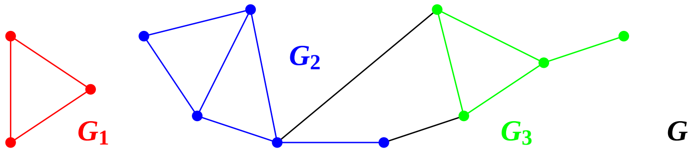
Definition

A graph H is an *induced subgraph* of G
$\iff V_H \subset V$, and $E_H = \{xy \in E : x, y \in V_H\}$.

Induced subgraphs are determined by their vertex sets.

Definition

A *partition* of G is a family of induced subgraphs $\{G_i = (V_i, E_i)\}_{i=1}^{l}$ of G such that $V = \sqcup_{i=1}^{l} V_i$ (disjoint union).



Higher order expander constant

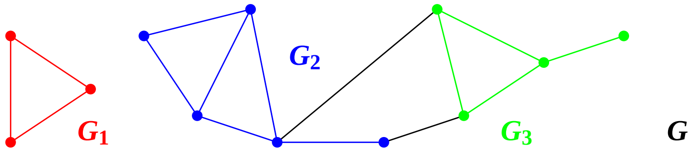
Definition

The *higher order expander constant* of G is

$$h_l(G) := \min \left\{ \max_{i=1,2,\dots,l} \frac{|\partial V_i|}{|V_i|} : \{G_i = (V_i, E_i)\}_{i=1}^{l} \text{ is a partition of } G \right\}$$

for each $l \in \mathbb{N}$.

In particular, $h(G) = h_2(G)$.



Example

$$h_l(K^{ln}) = n, \qquad h_3(G_{2n,2n}) = n, \qquad h_2(G_{2n,2n}) = \frac{1}{2n}$$

where $G_{2n,2n}$ was a graph constracted by connecting K^{2n} and K^{2n} by one edge.

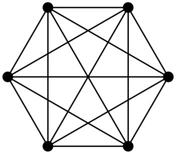


Figure: K^6

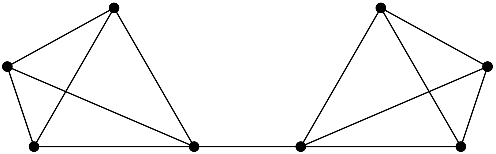


Figure: $G_{4,4}$

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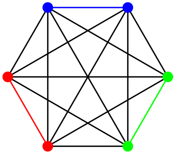

Figure: K^6

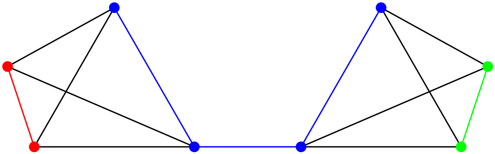

Figure: $G_{4,4}$

Theorem 1

$$\frac{\lambda_l(G)}{2l} \leq h_l(G) \leq 3^l \sqrt{2\deg(G)\lambda_l(G)} \tag{2}$$

for $l > 2$.

This shows the higher order expander constant $h_l(G)$ is estimated by $\lambda_l(G)$ as the inequalities (1),

$$\frac{\lambda_2(G)}{2} \leq h(G) \leq \sqrt{2\deg(G)\lambda_2(G)}.$$

This is also regarded as a numerical generalization of the fact

- The number of connected components of G is l
  $\Leftrightarrow \lambda_l(G) = 0$ and $\lambda_{l+1}(G) > 0$.

Theorem 2

1. *For any partition* $\{G_i\}_{i=1}^{l}$ *of* G

$$\min_{i=1,2,\dots,l} \lambda_2(G_i) \le \lambda_{l+1}(G). \tag{3}$$

2. *If for some* $l \in \mathbb{N}$

$$\lambda_{l+1}(G) > 2(l+1)3^{l+1}\sqrt{2\deg(G)\lambda_l(G)}, \tag{4}$$

then there exists a partition $\{G_i\}_{i=1}^{l}$ *of* G *such that*

$$\lambda_{l+1}(G) \le 2(l+1)3^{l+1}\min_{i=1,2,\dots,l} h(G_i). \tag{5}$$

This theorem means that $\lambda_{l+1}(G)$ represents strength of connection of each induced subgraph in a partition, if $\lambda_{l+1}(G) \gg \lambda_l(G)$.

Using (1), $\lambda_2(G)/2 \leq h(G)$ and (3), $\min_{i=1,2,\ldots,l} \lambda_2(G_i) \leq \lambda_{l+1}(G)$, we obtain

Example

If $l = 2$ and $n > 2 \cdot 3^4$, then $G_{2n,2n}$ satisfies the assumption (4).

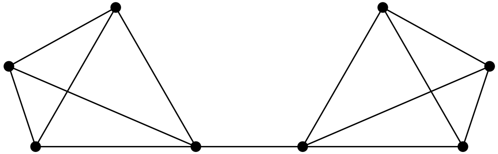

Figure: $G_{2n,2n}$

The assumption (4) seems to be a very strong condition.

Problem

Can we weaken the assumption (4)*?*

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