Higher eigenvalues of the Laplacian on a graph and partitions of the graph

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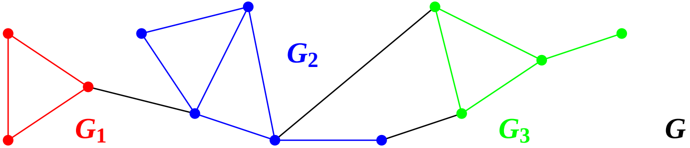
$$\lambda_l(G) \quad \longleftrightarrow \quad h_l(G) \quad \longleftrightarrow \quad \min_{i=1,2,\ldots,l} \lambda_2(G_i)$$

$\lambda_l(G)$: l-th eigenvalue of the Laplacian on a finite graph G
$h_l(G)$: the expander constant of G (a quantity of some connectivity)
$\{G_i\}_{i=1}^{l}$: a partition of G



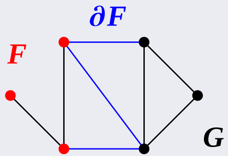
Expander constant

Assume graphs are non-oriented, finite, and don't have loops and multiple edges.
$G = (V, E)$: a graph

Definition

The *expander constant* of G is

$$h(G) = \min_{F \subset V} \left\{ \frac{|\partial F|}{|F|} : 1 \leq |F| \leq \frac{|V|}{2} \right\}$$



where ∂F is the set of the edges connecting F and $V - F$.

This represents strength of connection between two disjoint vertex sets.

Example

Let K^{2n}, K^{n_1}, K^{n_2} be complete graphs.
Let G_{n_1,n_2} be a graph with $V_{G_{n_1,n_2}} = V_{K^{n_1}} \cup V_{K^{n_2}}$ and $E_{G_{n_1,n_2}} = E_{K^{n_1}} \cup E_{K^{n_2}} \cup \{xy\}$ for some $x \in V_{K^{n_1}}$ and $y \in V_{K^{n_2}}$.
Then

$$h(K^{2n}) = n \geq 1, \quad h(G_{n_1,n_2}) = \frac{1}{\min\{n_1, n_2\}} \leq 1.$$

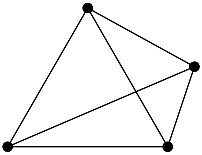


Figure: K^4

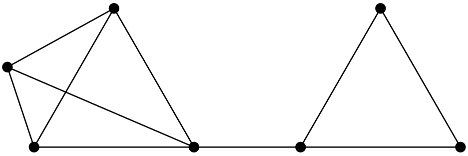


Figure: $G_{4,3}$

Eigenvalues of Laplacian

$\deg(x) := |\{y \in V : xy \in E\}|$ for $x \in V$.

Definition

The *Laplacian* Δ_G on G is a linear operator on $\mathbb{R}^V = \{f : V \to \mathbb{R}\}$ defined by

$$\Delta_G f(x) := f(x)\deg(x) - \sum_{xy \in E} f(y)$$

for $f \in \mathbb{R}^V$ and $x \in V$.

$\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_{|V|}(G)$: the eigenvalues of the Laplacian

- The number of connected components of G is l $\Leftrightarrow \lambda_l(G) = 0$ and $\lambda_{l+1}(G) > 0$.

Expander constant and Second eigenvalue of Laplacian

$\deg(G) := \max_{x \in V} \deg(x)$.

Theorem (Alon-Milman, Dodziuk)

$$\frac{\lambda_2(G)}{2} \leq h(G) \leq \sqrt{2 \deg(G) \lambda_2(G)}. \tag{1}$$

This theorem implies that the second eigenvalue $\lambda_2(G)$ also represents some strength of connection between two disjoint subgraphs.

Question

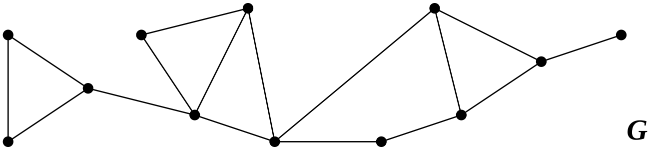
Question

For $l > 2$, can we relate $\lambda_l(G)$ to some connectivity of a graph and subgraphs as the inequalities (1)?

$$\lambda_l(G)$$

$$\updownarrow \; ?$$

Subgraphs of G



Partition

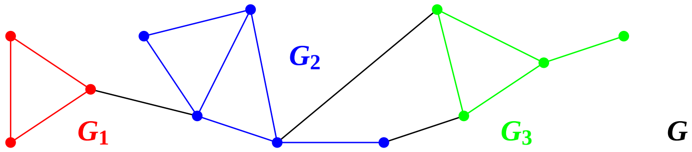
Definition

A graph H is an *induced subgraph* of G
$\iff V_H \subset V$, and $E_H = \{xy \in E : x, y \in V_H\}$.

Induced subgraphs are determined by their vertex sets.

Definition

A *partition* of G is a family of induced subgraphs $\{G_i = (V_i, E_i)\}_{i=1}^{l}$ of G such that $V = \sqcup_{i=1}^{l} V_i$ (disjoint union).



Higher order expander constant

Definition

The *higher order expander constant* of G is

$$h_l(G) := \min\left\{ \max_{i=1,2,\dots,l} \frac{|\partial V_i|}{|V_i|} : \{G_i = (V_i, E_i)\}_{i=1}^{l} \text{ is a partition of } G \right\}$$

for each $l \in \mathbb{N}$.

In particular, $h(G) = h_2(G)$.

Example

$$h_l(K^{ln}) = n, \qquad h_3(G_{2n,2n}) = n, \qquad h_2(G_{2n,2n}) = \frac{1}{2n}$$

where $G_{2n,2n}$ was a graph constracted by connecting K^{2n} and K^{2n} by one edge.

Theorem 1

$$\frac{\lambda_l(G)}{2l} \leq h_l(G) \leq 3^l \sqrt{2\deg(G)\lambda_l(G)} \tag{2}$$

for every $l \in \mathbb{N}$.

This is a generalization of the inequalities (1),

$$\frac{\lambda_2(G)}{2} \leq h(G) \leq \sqrt{2\deg(G)\lambda_2(G)}.$$

This is also regarded as a numerical generalization of the fact

- The number of connected components of G is l
  $\Leftrightarrow \lambda_l(G) = 0$ and $\lambda_{l+1}(G) > 0$.

Theorem 2

1. *For any partition* $\{G_i\}_{i=1}^{l}$ *of* G

$$\min_{i=1,2,\ldots,l} \lambda_2(G_i) \le \lambda_{l+1}(G). \tag{3}$$

2. *If for some* $l \in \mathbb{N}$

$$\lambda_{l+1}(G) > 2(l+1)3^{l+1}\sqrt{2\deg(G)\lambda_l(G)}, \tag{4}$$

then there exists a partition $\{G_i\}_{i=1}^{l}$ *of* G *such that*

$$\lambda_{l+1}(G) \le 2(l+1)3^{l+1} \min_{i=1,2,\ldots,l} h(G_i). \tag{5}$$

This theorem means that $\lambda_{l+1}(G)$ represents strength of connection of each induced subgraph in a partition, if $\lambda_{l+1}(G) \gg \lambda_l(G)$.

Using (1), $\lambda_2(G)/2 \le h(G)$ and (3), $\min_{i=1,2,\dots,l} \lambda_2(G_i) \le \lambda_{l+1}(G)$, we obtain the following example.

Example

If $l = 2$ and $n > 2 \cdot 3^4$, then $G_{2n,2n}$ satisfies the assumption (4), $\lambda_3(G_{2n,2n}) > 2 \cdot 3^4 \sqrt{2 \deg(G_{2n,2n}) \lambda_2(G_{2n,2n})}$.

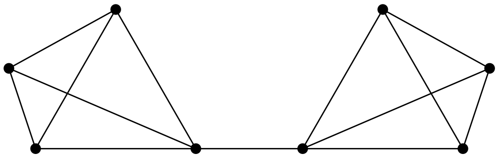


Figure: $G_{2n,2n}$

The assumption (4) seems to be a very strong condition.

Problem

Can we weaken the assumption (4)?

Outline of Proofs

- $\langle f, g\rangle := \sum_{x\in V} f(x)g(x)$, $||f|| := \sqrt{\langle f, f\rangle}$ for $f, g \in \mathbb{R}^V$
- $\langle a, b\rangle := \sum_{e\in V} a(e)b(e)$, $||a|| := \sqrt{\langle a, a\rangle}$ for $a, b \in \mathbb{R}^E$.
- $d : \mathbb{R}^V \to \mathbb{R}^E$, $df(xy) := |f(x) - f(y)|$ for $f \in \mathbb{R}^V$ and $xy \in E$.

Then $\langle \Delta_G f, f\rangle = ||df||^2$ for $f \in \mathbb{R}^V$, and consequently

$$\lambda_{l+1}(G) = \sup_{L_l} \inf_{f\in\mathbb{R}^V} \left\{ \frac{||df||^2}{||f||^2} : f \perp L_l, f \not\equiv 0 \right\} \quad (6)$$

$$\lambda_l(G) = \inf_{L_l} \sup_{f\in\mathbb{R}^V} \left\{ \frac{||df||^2}{||f||^2} : f \in L_l, f \not\equiv 0 \right\} \quad (7)$$

where L_l is an l-dimensional subspace of $\mathbb{R}^V$.

Key of Proofs

The keys of proofs of theorems are

- Choice of L_l in the inequalities (6) and (7),

$$\lambda_{l+1}(G) = \sup_{L_l} \inf_{f \in \mathbb{R}^V} \left\{ \frac{||df||^2}{||f||^2} : f \perp L_l, f \not\equiv 0 \right\} \qquad (6)$$

$$\lambda_l(G) = \inf_{L_l} \sup_{f \in \mathbb{R}^V} \left\{ \frac{||df||^2}{||f||^2} : f \in L_l, f \not\equiv 0 \right\}, \qquad (7)$$

- Choice of a partition of G.

Outline of Proof of $\lambda_l(G)/2l \le h_l(G)$ in (2)

For a partition $\{G_i = (V_i, E_i)\}_{i=1}^{l}$ of G, let $\psi_0 \equiv 1 \in \mathbb{R}^V$ and

$$\psi_i(x) = \begin{cases} |V_{i+1}| & \text{if } x \in \cup_{j=1}^{i} V_j \\ -\left|\cup_{j=1}^{i} V_j\right| & \text{if } x \in V_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, \ldots, l-1$. Then the subspace
$P := \langle \psi_0, \psi_1, \psi_2, \ldots, \psi_{l-1} \rangle$ of $\mathbb{R}^V$ is l-dimensional. Then

$$\lambda_l(G) \le \sup_{f \in \mathbb{R}^V} \left\{ \frac{||df||^2}{||f||^2} : f \in P, f \not\equiv 0 \right\} \le 2l \max_{i=1,2,\ldots,l} \frac{|\partial V_i|}{|V_i|},$$

by computing $||df||^2/||f||^2$ for $f \in P$. Hence $\lambda_l(G) \le 2l h_l(G)$.

Outline of Proof of (3), $\min_{i=1,2,\ldots,l} \lambda_2(G_i) \leq \lambda_{l+1}(G)$

For a partition $\{G_i = (V_i, E_i)\}_{i=1}^{l}$ of G, define $\psi_0, \psi_1, \psi_2, \ldots, \psi_{l-1}$ and $P = \langle \psi_0, \psi_1, \psi_2, \ldots, \psi_{l-1} \rangle$ as in the previous slide. Then for $f \in \mathbb{R}^V$ with $f \perp P$ we can show that

$$f|_{V_i} \perp \psi_0|_{V_i}$$

for all $i = 1, 2, \ldots, l-1$. Using this, we get

$$||df||^2 \geq \min_{i=1,2,\ldots,l} \lambda_2(G_i)||f||^2.$$

Hence

$$\lambda_{l+1}(G) \geq \inf_{f \in \mathbb{R}^V} \left\{ \frac{||df||^2}{||f||^2} : f \perp P, f \not\equiv 0 \right\} \geq \min_{i=1,2,\ldots,l} \lambda_2(G_i).$$

Outline of Proof of $h_l(G) \le 3^l \sqrt{2\deg(G)\lambda_l(G)}$ in (2)

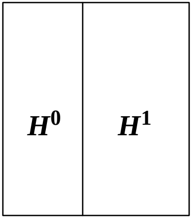
We construct a partition $\{G_i = (V_i, E_i)\}_{i=1}^{l}$ satisfying

$$h_l(G) \le \max_{i=1,2,\dots,l} \frac{|\partial V_i|}{|V_i|} \le 3^l \sqrt{2\deg(G)\lambda_l(G)}. \tag{8}$$

(1) Take an induced subgraph H^0 of G s.t.

$$\frac{|\partial V_{H^0}|}{|V_{H^0}|} = h(G) \quad \text{and} \quad |V_{H^0}| \le \frac{|V|}{2},$$

and set $H^1 := G - H^0$, where $G - H^0$ means the induced subgraph of G whose vertex set is $V - V_{H^0}$,



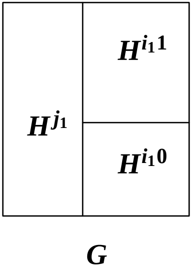
G

Outline of Proof of $h_l(G) \leq 3^l \sqrt{2\deg(G)\lambda_l(G)}$ in (2)

(2) Assume $h(H^{i_1}) \leq h(H^{j_1})$ where $i_1, j_1 \in \{0,1\}$ and $i_1 \neq j_1$. Then take an induced subgraph $H^{i_1 0}$ of H^{i_1} s.t.

$$\frac{|\partial_{H^{i_1}} V_{H^{i_1 0}}|}{|V_{H^{i_1 0}}|} = h(H^{i_1}) \quad \text{and} \quad |V_{H^{i_1 0}}| \leq \frac{|V_{H^{i_1}}|}{2},$$

and set $H^{i_1 1} := H^{i_1} - H^{i_1 0}$, where $\partial_{G_1} V_{G_2} := \{xy \in E_{G_1} | x \in V_{G_2}, y \in V_{G_1 - G_2}\}$ for graphs $G_2 \subset G_1$.

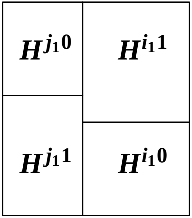


Outline of Proof of $h_l(G) \leq 3^l \sqrt{2\deg(G)\lambda_l(G)}$ in (2)

(3-1) If $h(H^{j_1}) \leq \min\{h(H^{i_1 0}), h(H^{i_1 1})\}$, then take a subgraph $H^{j_1 0}$ of H^{j_1} s.t.

$$\frac{|\partial_{H^{j_1}} V_{H^{j_1 0}}|}{|V_{H^{j_1 0}}|} = h(H^{j_1}) \quad \text{and} \quad |V_{H^{j_1 0}}| \leq \frac{|V_{H^{j_1}}|}{2},$$

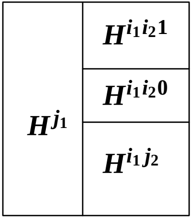
and set $H^{j_1 1} := H^{j_1} - H^{j_1 0}$.



(3-2) If $\min\{h(H^{i_1 0}), h(H^{i_1 1})\} < h(H^{j})$, then set $h(H^{i_1 i_2}) \leq h(H^{i_1 j_2})$ where $i_2, j_2 \in \{0, 1\}$ and $i_2 \neq j_2$. Take a subgraph $H^{i_1 i_2 0}$ of $H^{i_1 i_2}$ s.t.

$$\frac{|\partial_{H^{i_1 i_2}} V_{H^{i_1 i_2 0}}|}{|V_{H^{i_1 i_2 0}}|} = h(H^{i_1 i_2}) \text{ and } |V_{H^{i_1 i_2 0}}| \leq \frac{|V_{H^{i_1 i_2}}|}{2},$$

and set $H^{i_1 i_2 1} := H^{i_1 i_2} - H^{i_1 i_2 0}$.

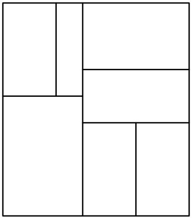


Outline of Proof of $h_l(G) \leq 3^l \sqrt{2\deg(G)\lambda_l(G)}$ in (2)

Inductively, we divide an undivided subgraph in $\{H^{a_1 a_2 \cdots a_m}\}$ with the minimum expanding constant, into a subgraph which attains the expander constant and the complement subgraph.

Repeat this procedure until the number of the undivided subgraphs in $\{H^{a_1 a_2 \cdots a_m}\}$ becomes l.

Consequently we divided G into l subgraphs.



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Let $\{G_i = (V_i, E_i)\}_{i=1}^{l}$ be the set of the undivided subgraphs in $\{H^{a_1 a_2 \cdots a_m}\}$. Then $\{G_i\}_{i=1}^{l}$ is a partition, and we can prove (8),

$$h_l(G) \leq \max_{i=1,2,\ldots,l} \frac{|\partial V_i|}{|V_i|} \leq 3^l \sqrt{2\deg(G)\lambda_l(G)}.$$

Outline of Proof of (5), $\lambda_{l+1}(G) \le 2(l+1)3^{l+1} \min_{i=1,2,\dots,l} h(G_i)$

Let $\{\tilde{G}_i = (\tilde{V}_i, \tilde{E}_i)\}_{i=1}^{l+1}$ be a partition constructed by deviding $\{G_i = (V_i, E_i)\}_{i=1}^{l}$ in the previous slide once more as the previous procedure. Then

$$\max_{i=1,2,\dots,l+1} \frac{|\partial \tilde{V}_i|}{|\tilde{V}_i|} \le 3^{l+1} \max\left\{ \sqrt{2\deg(G)\lambda_l(G)}, \min_{i=1,2,\dots,l} h(G_i) \right\}.$$

Using (2) and the assumption in 2 in Theorem 2, we have

$$\begin{aligned}
\lambda_{l+1}(G) &\le 2(l+1)h_{l+1}(G) \\
&\le 2(l+1) \max_{i=1,2,\dots,l+1} \frac{|\partial \tilde{V}_i|}{|\tilde{V}_i|} \\
&\le 2(l+1)3^{l+1} \max\left\{ \sqrt{2\deg(G)\lambda_l(G)}, \min_{i=1,2,\dots,l} h(G_i) \right\} \\
&\le 2(l+1)3^{l+1} \min_{i=1,2,\dots,l} h(G_i).
\end{aligned}$$

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