Multi-way expansion constants and expander graphs

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We assume graphs are non-oriented, don't have loops and multiple edges. Let $G = (V, E)$ be a finite graph. Lee-Gharan-Trevisan [LGT] define the *k-way expansion constant* of G for $1 \leq k \leq |V|$ as

$$h_k(G) := \min \left\{ \max_{i=1,2,\ldots,k} \frac{|\partial V^i|}{|V^i|} : V = \bigsqcup_{i=1}^{k} V^i \text{ (disjoint union)}, V^i \neq \emptyset \right\}$$

where $\partial V^i := \{vw \in E : v \in V^i, w \in V - V^i\}$. The 2-way expansion constant is well-known as the expansion constant (or the isoperimetric constant). The k-way expansion constant represents strength of connection between k disjoint subgraphs in the graph. In particular, $h_k(G) = 0$ and $h_{k+1}(G) > 0$ if and only if the number of connected components of G is k. In this case, $h_{k+1}(G) = \min_{i=1,2,\ldots,k} h_2(G^i)$ where G^i's are connected components of G.

An *induced subgraph* of $G = (V, E)$ is a subgraph $H = (V_H, E_H)$ satisfying $V_H \subset V$ and $E_H = \{xy \in E : x, y \in V_H\}$. A *k-partition* of G is a family of induced subgraphs $\{G^i = (V^i, E^i)\}_{i=1}^k$ such that $V = \sqcup_{i=1}^k V^i$. We can easily show

Lemma 1. *For any k-partition* $\{G^i = (V^i, E^i)\}_{i=1}^k$ *of* G

$$h_{k+1}(G) \geq \min_{i=1,2,\ldots,k} h_2(G^i).$$

On the other hand, we have

Theorem 2. *If* $h_{k+1}(G) > 3^{k+1} h_k(G)$ *for some* $k \in \mathbb{N}$*, then there exists a k-partition* $\{G^i = (V^i, E^i)\}_{i=1}^k$ *of* G *satisfying*

$$\frac{h_{k+1}(G)}{3^{k+1}} \leq \min_{i=1,2,\ldots,k} h_2(G^i), \quad \max_{i=1,2,\ldots,k} \frac{|\partial V_i|}{|V_i|} \leq 3^k h_k(G).$$

Let $\deg(x)$ denotes the *degree* at $x \in V$, which is the number of edges such that x is an end point of it, and $\deg(G) := \max_{x \in V} \deg(x)$. Then for the eigenvalues of the (normalized) Laplacian on a graph, we can obtain similar theorem, using the result by Lee-Gharan-Trevisan [LGT], there is a constant $C > 0$ such that

$$\frac{\lambda_k(G)}{2\deg(G)} \leq h_k(G) \leq Ck^2 \deg(G) \sqrt{\lambda_k(G)} \tag{1}$$

for every connected graph G and every k, where $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_{|V|}(G)$ are the eigenvalues of the Laplacian. (Note that the original statement is described using the normalized Laplacian on a graph and it don't need the assumption that the graph is connected, and we can omit the dependence on $\deg(G)$.) On the other hand, we can also easily show that $\min_{i=1,2,\ldots,k} \lambda_2(G^i) \leq \lambda_{k+1}(G)$ for any k-partition $\{G^i\}_{i=1}^k$ of G.

A sequence of expander graphs is a sequence of finite graphs such that (i) $|V_n| \to \infty$ as $n \to \infty$; (ii) $\sup_{n\in\mathbb{N}} \deg(G_n) < \infty$; (iii) $\inf_{n\in\mathbb{N}} h_2(G_n) > 0$.

Corollary 3. *Let $\{G_n = (V_n, E_n)\}_{n=1}^\infty$ be a sequence of connected finite graphs such that* (i) *$|V_n| \to \infty$ as $n \to \infty$;* (ii) *$\sup_{n\in\mathbb{N}} \deg(G_n) < \infty$;* (iii) *$\inf_{n\in\mathbb{N}} h_{k'+1}(G_n) > 0$ for some $k' \in \mathbb{N}$. Then there are an increase sequence of numbers $\{n_m\}_{m=1}^\infty$ and $k \in \mathbb{N}$, k-partitions $\{H_m^i\}_{i=1}^k$ of G_{n_m} for all m such that $\{H_{n_m}^i\}_{m=1}^\infty$ are sequences of expanders for all $i = 1, 2, \ldots, k$.*

A similar result for Riemannian manifolds was given by Funano and Shioya [FS]: A sequence of closed weighted Riemannian manifolds whose k-th eigenvalues diverges to ∞ for a natural number k is a union of k Lévy families.

We can endow a graph G with the path metric $d_G(x, y)$ between vertices x and y which is the minimum number of edges connecting x and y. A sequence of metric spaces $\{X_n\}_{n=1}^\infty$ is said to be *coarsely embeddable* into a metric space Y if there exist two non-decreasing functions ρ_1 and ρ_2 on $[0, +\infty)$ and maps $f_n : X_n \to Y$ $(n = 1, 2, \ldots)$ such that (1) $\lim_{r\to\infty} \rho_1(r) = +\infty$; (2) $\rho_1(d_{X_n}(x, y)) \leq d_Y(f_n(x), f_n(y)) \leq \rho_2(d_{X_n}(x, y))$ for all $x, y \in X_n$ and n. This notion was defined by Gromov [Gro93], and he proved that a sequence of expander graphs is not coarsely embeddable into any Hilbert space. We can generalize this fact as follows:

Corollary 4. *Let $\{G_n = (V_n, E_n)\}_{n=1}^\infty$ be a sequence of finite graphs such that* (i) *$|V_n| \to \infty$ as $n \to \infty$;* (ii) *$\sup_{n\in\mathbb{N}} \deg(G_n) < \infty$;* (iii) *$\inf_{n\in\mathbb{N}} h_{k+1}(G_n) > 0$ for some $k \in \mathbb{N}$. Then $\{G_n = (V_n, E_n)\}_{n=1}^\infty$ is not coarsely embeddable into any Hilbert space.*

References

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