Multi-way expansion constants and expander graphs

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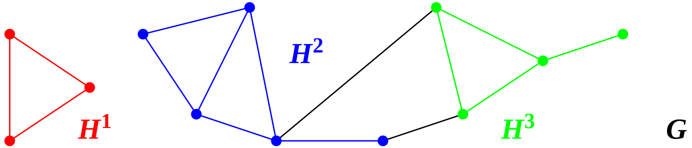
Overview

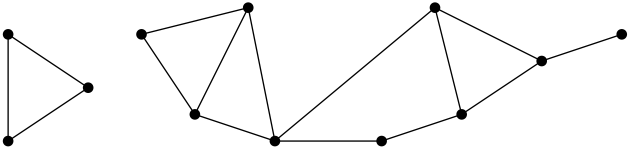
- Give a relation between multi-way expansion constants $h_k(G)$, $h_{k+1}(G)$ of a graph G, which represent strength of connectivity of G, and a k-partition $\{H^i\}_{i=1}^k$ of G



- Using a relation between $h_k(G)$ and the k-th eigenvalue $\lambda_k(G)$ of the Laplacian on G, review the above relation
- See the coarse non-embeddability of a sequence of "generalized" expander graphs into Hilbert spaces

Graph

A graph $G = (V, E)$ is a pair of a vertex set V and an edge set $E \subset \{vw : v, w \in V\}$.



G

Assume graphs are finite ($|V| < \infty$), undirected ($vw = wv$), and without loops ($vv \notin E$).

$\deg(v) := |\{w \in V : vw \in E\}|$ for $v \in V$.
$\deg(G) := \max_{v \in V} \deg(v)$.

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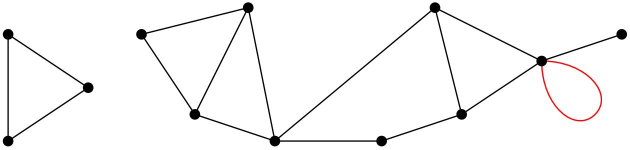

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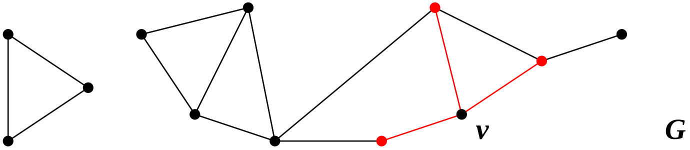

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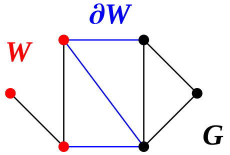
Expansion constant $h(G)$

Definition (Expansion (Isoperimetric, Cheeger) consatnt)

The *expansion constant* of $G = (V, E)$ with $|V| \geq 2$ is defined as

$$h(G) = \min_{\emptyset \neq W \subset V} \left\{ \max \left\{ \frac{|\partial W|}{|W|}, \frac{|\partial W|}{|V - W|} \right\} \right\}.$$



Here $\partial W := \{vw : v \in W,\ w \in V - W\}$.
Note that $h(G) > 0$ if and only if G is connected.
For $\epsilon > 0$, a graph G with $h(G) \geq \epsilon$ is called an ϵ-*expander graph*.

Example

- For $n \in \mathbb{N}$, $h(K^{2n}) = n$.
- Let $G_{n_1,n_2} := (V_{K^{n_1}} \cup V_{K^{n_2}}, E_{K^{n_1}} \cup E_{K^{n_2}} \cup \{vw\})$ for $n_1, n_2 \in \mathbb{N}$, where $v \in V_{K^{n_1}}$, $w \in V_{K^{n_2}}$. Then $h(G_{n_1,n_2}) = 1/\min\{n_1, n_2\}$.

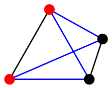


Figure: K^4

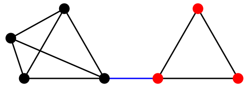


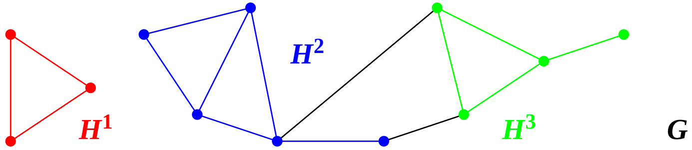
Figure: $G_{4,3}$

Partition of a graph

A graph $H = (V_H, E_H)$ is said to be an *induced subgraph* of G if $V_H \subset V$ and $E_H = \{xy \in E : x, y \in V_H\}$.
Induced subgraphs of G are determined by their vertices.

Definition (Partition of a graph)

A k-*partition* of G is a family $\{H^i = (V^i, E^i)\}_{i=1}^k$ of induced subgraphs of G satisfying $V = \sqcup_{i=1}^k V^i$ (disjoint union) and $V^i \neq \emptyset$ for all i.



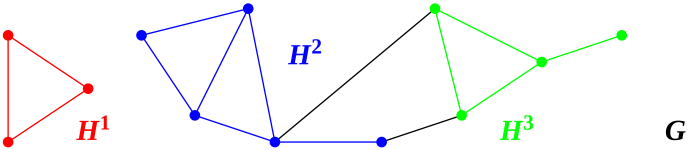
Multi-way expansion constant $h_k(G)$

Definition (Lee-Gharan-Trevisan)

For $k \in \mathbb{N}$, the k-*way expansion constant* of $G = (V, E)$ with $|V| \geq k$ is defined as

$$h_k(G) := \min\left\{ \max_{i=1,2,\dots,k} \frac{|\partial V^i|}{|V^i|} : \{H^i = (V^i, E^i)\}_{i=1}^k \text{ is a } k\text{-partition} \right\}$$

for $k \in \{1, 2, \dots, n\}$. In particular, $h_2(G) = h(G)$.



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Example

- For $n, k \in \mathbb{N}$, $h_k(K^{kn}) = (k-1)n$.
- For $n \in \mathbb{N}$, $h_3(G_{2n,2n}) = n$, $h_2(G_{2n,2n}) = 1/2n$.

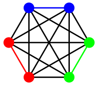

Figure: K^6

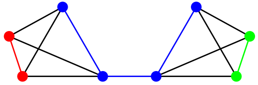

Figure: $G_{4,4}$

Some properties of $h_k(G)$

- $0 = h_1(G) \leq h_2(G) \leq h_3(G) \leq \cdots \leq h_n(G) \leq \deg(G)$.

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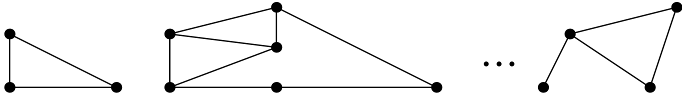
- $h_k(G) = 0$ and $h_{k+1}(G) > 0$ if and only if the number of the connected components of G is k.



H^1 H^2 H^k

In this situation if each connected components $H^i = (V^i, E^i)$ of G satisfies $|V^i| \ge 2$, then

$$h_{k+1}(G) = \min_{i=1,2,\ldots,k} h(H^i).$$

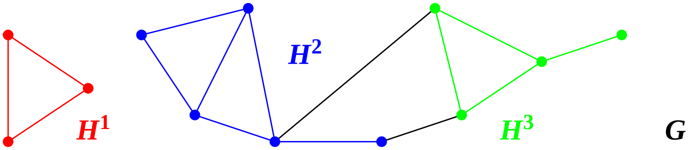
In particular, $h(G) > 0$ if and only if G is connected.

Connectivity of induced subgraphs of a partition

Lemma

For any k-partition $\{H^i = (V^i, E^i)\}_{i=1}^k$ of G with $|V^i| \geq 2$, we have

$$h_{k+1}(G) \geq \min_{i=1,2,\ldots,k} h(H^i).$$



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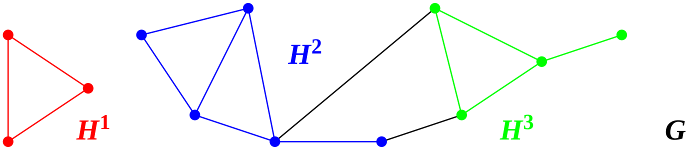


Theorem

If $h_{k+1}(G)/3^{k+1} > h_k(G)$ for some $k \in \mathbb{N}$, then there exists a k-partition $\{H^i = (V^i, E^i)\}_{i=1}^k$ of G satisfying

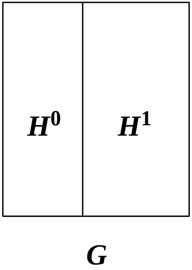
$$\frac{h_{k+1}(G)}{3^{k+1}} \leq \min_{i=1,2,\ldots,k} h(H^i), \quad \max_{i=1,2,\ldots,k} \frac{|\partial V_i|}{|V_i|} \leq 3^k h_k(G).$$

Construction of the partition in Theorem 1

(1) Take an induced subgraph H^0 of G s.t.

$$\frac{|\partial V_{H^0}|}{|V_{H^0}|} = h(G) \quad \text{and} \quad |V_{H^0}| \le \frac{|V|}{2},$$

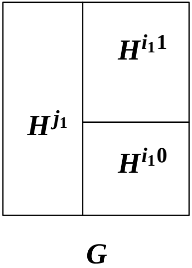
and set $H^1 := G - H^0$, where $G - H^0$ means the induced subgraph of G whose vertex set is $V - V_{H^0}$,



(2) Assume $h(H^{i_1}) \le h(H^{j_1})$ where $i_1, j_1 \in \{0,1\}$ and $i_1 \neq j_1$. Then take an induced subgraph $H^{i_1 0}$ of H^{i_1} s.t.

$$\frac{|\partial_{H^{i_1}} V_{H^{i_1 0}}|}{|V_{H^{i_1 0}}|} = h(H^{i_1}) \quad \text{and} \quad |V_{H^{i_1 0}}| \le \frac{|V_{H^{i_1}}|}{2},$$

and set $H^{i_1 1} := H^{i_1} - H^{i_1 0}$, where $\partial_{G_1} V_{G_2} := \{xy \in E_{G_1} | x \in V_{G_2}, y \in V_{G_1 - G_2}\}$ for graphs $G_2 \subset G_1$.

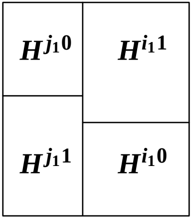


Construction of the partition in Theorem 1

(3-1) If $h(H^{j_1}) \leq \min\{h(H^{i_1 0}), h(H^{i_1 1})\}$, then take a subgraph $H^{j_1 0}$ of H^{j_1} s.t.

$$\frac{|\partial_{H^{j_1}} V_{H^{j_1 0}}|}{|V_{H^{j_1 0}}|} = h(H^{j_1}) \quad \text{and} \quad |V_{H^{j_1 0}}| \leq \frac{|V_{H^{j_1}}|}{2},$$

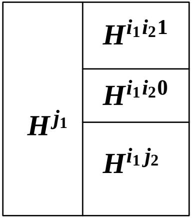
and set $H^{j_1 1} := H^{j_1} - H^{j_1 0}$.



(3-2) If $\min\{h(H^{i_1 0}), h(H^{i_1 1})\} < h(H^{j})$, then set $h(H^{i_1 i_2}) \leq h(H^{i_1 j_2})$ where $i_2, j_2 \in \{0, 1\}$ and $i_2 \neq j_2$. Take a subgraph $H^{i_1 i_2 0}$ of $H^{i_1 i_2}$ s.t.

$$\frac{|\partial_{H^{i_1 i_2}} V_{H^{i_1 i_2 0}}|}{|V_{H^{i_1 i_2 0}}|} = h(H^{i_1 i_2}) \text{ and } |V_{H^{i_1 i_2 0}}| \leq \frac{|V_{H^{i_1 i_2}}|}{2},$$

and set $H^{i_1 i_2 1} := H^{i_1 i_2} - H^{i_1 i_2 0}$.



G

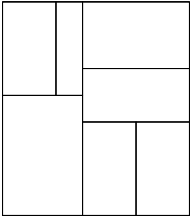
Construction of the partition in Theorem 1

Inductively, we divide an undivided subgraph in $\{H^{a_1 a_2 \cdots a_m}\}$ with the minimum expanding constant, into a subgraph which attains the expander constant and the complement subgraph.

Repeat this procedure until the number of the undivided subgraphs in $\{H^{a_1 a_2 \cdots a_m}\}$ becomes k.

Consequently we divided G into k subgraphs.

Then it is a partition in Theorem 1.



G

A sequence of expander graphs

A *sequence of expander graphs* is a sequence of finite graphs $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ such that

(i) $\lim_{n\to\infty} |V_n| = \infty$;

(ii) $\sup_{n\in\mathbb{N}} \deg(G_n) < \infty$;

(iii) $\inf_{n\in\mathbb{N}} h(G_n) > 0$.

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Corollary

Let $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ be a sequence of finite graphs such that

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(ii) $\sup_{n\in\mathbb{N}} \deg(G_n) < \infty$;

(iii)' $\inf_{n\in\mathbb{N}} h_{k'+1}(G_n) > 0$ *for some* $k' \in \mathbb{N}$.

Then there are $k \in \mathbb{N}$ *with* $k \leq k'$, *subsequence* $\{G_m\}_{m=1}^{\infty}$ *of* $\{G_n\}_{n=1}^{\infty}$, *and* k*-partitions* $\left\{H_m^i\right\}_{i=1}^{k}$ *of* G_m *for each* m *such that* $\left\{H_m^i\right\}_{m=1}^{\infty}$ *are sequences of expanders for each* $i = 1, 2, \ldots, k$.

Eigenvalues $\lambda_k(G)$ of the Laplacian of a graph

Let $G = (V, E)$ be a graph, and $V = \{v_1, v_2, \ldots, v_n\}$.
The *Laplacian* on G is an $n \times n$ symmetric integer matrix
$\Delta_G := D(G) - A(G)$, where

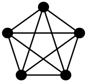
$$D(G)_{ij} := \begin{cases} \deg(v_i) & (i = j) \\ 0 & \text{(otherwise)}, \end{cases} \qquad A(G)_{ij} := \begin{cases} 1 & (v_i v_j \in E) \\ 0 & \text{(otherwise)}. \end{cases}$$

Let $\lambda_1(G) \le \lambda_2(G) \le \cdots \le \lambda_n(G)$ be the eigenvalues of Δ_G.

Example

- Complete graphs

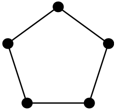
  $\lambda_1(K^n) = 0$, $\lambda_k(K^n) = n$ for $k = 2, 3, \ldots, n$;



K^5

- Cycles

  $C^n := (\{1, 2, \ldots, n\}, \{vw; |v - w| = 1 \mod n\})$
  $\{\lambda_k(C^n)\}_{k=1}^n = \{2 - 2\cos(2\pi(j-1)/n)\}_{j=1}^n$.



C^5

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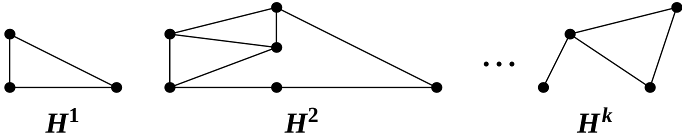

In this situation if each connected components H^i of G satisfy $|V^i| \ge 2$, then

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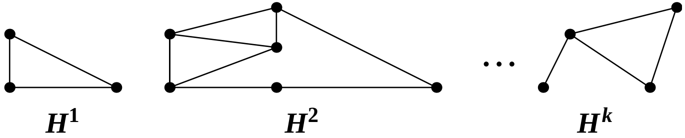

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As above $\lambda_k(G)$ and $h_k(G)$ have similar properties.

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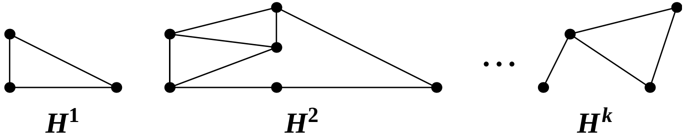

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Lemma

For any k-partition $\{H^i\}_{i=1}^k$ of G with $|V^i| \geq 2$, we have

$$\lambda_{k+1}(G) \geq \min_{i=1,2,\ldots,k} \lambda_2(H^i).$$

Relation between $\lambda_k(G)$ and $h_k(G)$

Theorem (Lee-Gharan-Trevisan ’12)

There is $C > 0$ such that for any connected graph $G = (V, E)$ with $|V| \geq k$

$$\frac{\lambda_k(G)}{2\deg(G)} \leq h_k(G) \leq Ck^2 \deg(G)\sqrt{\lambda_k(G)}.$$

In reality, they prove similar inequalities, whose coefficients don’t depends on $\deg(G)$, for the weighted multi-way expansion constant and eigenvalues of the normalized Laplacian on a graph G.

Relation between $\lambda_k(G)$ and $h_k(G)$

Corollary

Let $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ *be a sequence of finite graphs such that*

(i) $\lim_{n\to\infty} |V_n| = \infty$;

(ii) $\sup_{n\in\mathbb{N}} \deg(G_n) < \infty$;

(iii)' $\inf_{n\in\mathbb{N}} \lambda_{k'+1}(G_n) > 0$ *for some* $k' \in \mathbb{N}$.

Then there are $k \in \mathbb{N}$ *with* $k \leq k'$, *subsequence* $\{G_m\}_{m=1}^{\infty}$ *of* $\{G_n\}_{n=1}^{\infty}$, *and* k*-partitions* $\left\{H_m^i\right\}_{i=1}^{k}$ *of* G_m *for each* m *such that* $\left\{H_m^i\right\}_{m=1}^{\infty}$ *are sequences of expanders for each* $i = 1, 2, \ldots, k$.

This is a graph analog of the following result:

Theorem (Funano-Shioya)

A sequence of closed weighted Riemannian manifolds whose $(k+1)$*-st eigenvalues diverges to* ∞ *for a fixed natural number* k *is a union of* k *Lévy families.*

Coarse non-embeddability

Definition (Gromov)

A sequence of metric spaces $\{(X_n, d_{X_n})\}_{n=1}^{\infty}$ is said to be *coarsely embeddable* into a metric space (Y, d_Y) if there exist two non-decreasing functions $\rho_1, \rho_2 : [0, +\infty) \to [0, +\infty)$ and maps $f_n : X_n \to Y$ $(n = 1, 2, \dots)$ such that

(1) $\lim_{r\to\infty} \rho_1(r) = +\infty$;

(2) $\rho_1(d_{X_n}(x, y)) \leq d_Y(f_n(x), f_n(y)) \leq \rho_2(d_{X_n}(x, y))$ for all $x, y \in X_n$ and n.

We can endow a graph G with the path metric $d_G(x, y)$ between vertices x and y which is the minimum number of edges connecting x and y.

Theorem (Gromov)

A sequence of expander graphs is not coarsely embeddable into any Hilbert space.

Coarse non-embeddability

Proposition

Let $\{G_n\}_{n=1}^{\infty}$ *be a sequence of graphs with* $\sup_{n\in\mathbb{N}} \deg(G_n) < \infty$*. If there are induced subgraphs* H_n *of* G_n *for all* n *such that* $\{H_n\}_{n=1}^{\infty}$ *is a sequence of expanders, then* $\{G_n\}_{n=1}^{\infty}$ *is not coarsely embeddable into any Hilbert space.*

Hence we have the following:

Corollary

Let $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ *be a sequence of finite graphs such that*

(i) $\lim_{n\to\infty} |V_n| = \infty$;

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Then $\{G_n\}_{n=1}^{\infty}$ *is not coarsely embeddable into any Hilbert space.*

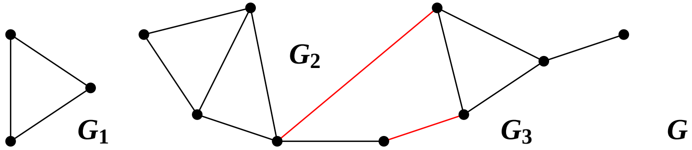
Thank you for your attention

Comment on the relation between $h_k(G)$ and $\lambda_k(G)$

Let $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$ be the eigenvectors with $||x_l|| = 1$ associated to $\lambda_1(G), \lambda_2(G), \ldots, \lambda_k(G)$ respectively. Then we can write

$$\lambda_l(G) = \langle \Delta_G x_l, x_l \rangle = \sum_{v_i v_j \in E} |(x_l)_i - (x_l)_j|^2.$$

Very roughly speaking, the k-partition $\{G_i = (V_i, E_i)\}_{i=1}^k$ which attain $h_k(G)$ is obtained by dividing G along edges $\{v_i v_j\}$ which $|(x_l)_i - (x_l)_j|$ are larger than the others for each $l = 1, 2, \ldots, k$.



Random graph

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For $n, d \in \mathbb{N}$, let $P_{n,d}$ be the uniform distribution on $\Pi_{n,d} := \left\{ \{\pi_j\}_{j=1}^d : \pi_j \text{ are permutations on } \{1, 2, \dots, n\} \right\}$.

For $\pi = \{\pi_j\}_{j=1}^d \in \Pi_{n,d}$, we define the graph

$$G(\pi) := \left(\{1, 2, \dots, n\}, \{i\pi_j(i) : 1 \le i \le n, 1 \le j \le d, i \ne \pi_j(i)\} \right).$$

Then $\deg(G(\pi)) \le 2d$.

Theorem (cf. Friedman '03)

For $d \ge 2$ there is a positive constant C such that

$$\lim_{n \to \infty} P_{n,d} \left(\left\{ \pi \in \Pi_{n,d} : \lambda_2 (G(\pi)) > C \right\} \right) = 1.$$

(*Moreover, he gave the best possible constant of C and the order for $2d$-regular graphs which allows loops and multiple edges.*)