p-Laplacian on finitely generated groups

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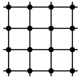
Motivation

In Mathematics, symmetry of structures is represented by groups. The study of properties of groups is important. In this talk, we consider properties of a finitely generated group geometrically.

$$\begin{array}{ccc} \text{Finitely generated group } \Gamma & \rightsquigarrow & \text{Cayley graph } G \text{ of } \Gamma \\ \wr\downarrow & & \wr\downarrow \\ \text{Property of } \Gamma & \leftrightsquigarrow & \text{Property of } G \end{array}$$



Our theorem has the form of

" Γ has a group property $\Longleftrightarrow$ G has a graph property ".

Finitely generated groups

Definition (Finitely generated group)

A group Γ is finitely generated
$:\Longleftrightarrow \exists$ a finite subset $S \subset \Gamma$ such that
$\forall \gamma \in \Gamma,\ \gamma = s_1 s_2 \cdots s_k$ for $\exists s_1, s_2, \ldots, s_k \in S$.

In this talk, we consider only finitely generated infinite groups.

Example

Free abelian groups $\mathbb{Z}^n$, free non-abelian groups F_n, $SL(n, \mathbb{Z})$, etc.

Property $(F_{\ell^p(\Gamma)})$

Γ : finitely generated group with a finite generating subset S

$$\ell^p(\Gamma) := \left\{ f : \Gamma \to \mathbb{R} \,\middle|\, \|f\|_p := \sum_{\gamma\in\Gamma} |f(\gamma)|^p < \infty \right\} \quad (p > 1)$$

Definition (Property $(F_{\ell^p(\Gamma)})$)

For $p > 1$, Γ has Property $(F_{\ell^p(\Gamma)})$

$:\Longleftrightarrow$ $\forall$ affine isometric action $\alpha : \Gamma \curvearrowright \ell^p(\Gamma)$ has a fixed point.

has a fixed point;

$\exists f_0 \in \ell^p(\Gamma)$ such that $\alpha(\gamma, f_0) = f_0$ for $\forall\gamma \in \Gamma$.

Known Examples and Result

Example (Groups with Property $(F_{\ell^p(\Gamma)})$ for all $p > 1$)

$SL(n, \mathbb{Z})$ $(n \geq 3)$

Example (Groups without Property $(F_{\ell^2(\Gamma)})$)

$SL(2, \mathbb{Z})$, Amenable groups

Example (Groups without Property $(F_{\ell^p(\Gamma)})$ for all $p > 1$)

Free abelian groups $\mathbb{Z}^n$, Free non-abelian groups F_n

Fact (cf. Yu '05, Bourdon-Martin-Valette '05)

$\exists\Gamma$ *with Property* (F_{ℓ^2}) *and without Property* (F_{ℓ^p}) *for a large* $p > 2$.

Property $(F_{\ell^p(\Gamma)})$

In this talk, we

- consider Property $(F_{\ell^p(\Gamma)})$ for an affine isometric action with $\lambda_{\Gamma,p}$ as the linear part,
- give one partial characterization of it using a graph property.

Here, the left regular representation

$$\lambda_{\Gamma,p} : \Gamma \curvearrowright \ell^p(\Gamma) \; ; \; \lambda_{\Gamma,p}(\gamma, f) := f(\gamma \cdot)$$

for $f \in \ell^p(\Gamma)$, $\gamma \in \Gamma$.

Cayley graphs

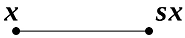
Γ : a finitely generated infinite group

S : a finite generating set of Γ s.t. if $s \in S$ then $s^{-1} \in S$, and $id \notin S$

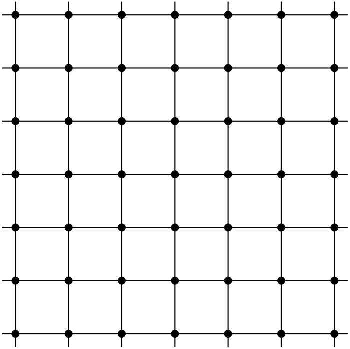
Definition (Cayley graph)

The graph $G = (\Gamma, E)$ is the Cayley graph of Γ, where

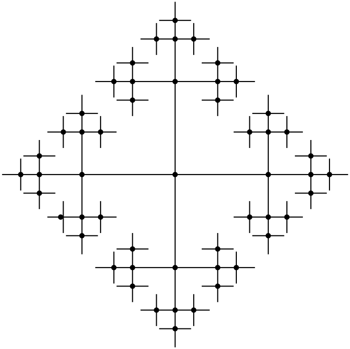
- Γ: the vertex set,
- $E := \{\{x, sx\} \subset V \mid x \in \Gamma, s \in S\}$: the edge set.



Example of Cayley graphs



$(\mathbb{Z}^2, +)$; $S = \{(1,0), (-1,0), (0,1), (0,-1)\}$

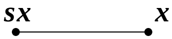


F_2; $S = \{s_1, s_1^{-1}, s_2, s_2^{-1}\}$

p-Dirichlet finite function

- For $f : \Gamma \to \mathbb{R}$, $x \in \Gamma$ and $s \in S$, define

$$d_s f(x) := f(sx) - f(x)\text{: the difference of } f.$$



- $D_p(\Gamma) := \{ f : \Gamma \to \mathbb{R} \mid d_s f \in \ell^p(\Gamma), \forall s \in S \} \supset \ell^p(\Gamma)$

$$\|f\|_{D_p(\Gamma)} := \left(\frac{1}{|S|} \sum_{s \in S} \|d_s f\|_p \right)^{1/p} \text{: semi-norm on } D_p(\Gamma)$$

The elements in $D_p(\Gamma)$ are called p-Dirichlet finite functions.

p-Laplacian

Definition

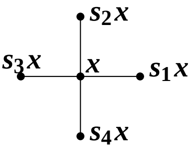
The p-Laplacian $\Delta_p : D_p(\Gamma) \to \ell^q(\Gamma)$ $(q = p/(p-1))$ is defined by

$$\Delta_p f(x) := \frac{1}{|S|} \sum_{s \in S} |d_s f(x)|^{p-2}(d_s f(x)),$$

where, if $p < 2$ and $d_s f(x) = 0$, we set $|d_s f(x)|^{p-2} = 0$.

If $p = 2$,

$$\Delta_2 f(x) := \frac{1}{|S|} \sum_{s \in S} f(sx) - f(x).$$



Theorem

Theorem (T.)

Let $p > 1$. The following are equivalent:

(i) *Every affine isometric action α of Γ on $\ell^p(\Gamma)$ with $\lambda_{\Gamma,p}$ as the linear part has a fixed point.*

(ii) *$\exists C > 0$ such that $\forall f \in D_p(\Gamma)$ satisfies*

$$\|\Delta_p f\|_q \geq C \|f\|_{D_p(\Gamma)}^{p-1},$$

where $q = p/(p-1)$.

Note for Theorem

When $p = 2$, the condition (ii) is

(ii)$_2$ $\exists C > 0$ such that $\forall f \in D_2(\Gamma)$ satisfies $\|\Delta_2 f\|_2 \geq C\|f\|_{D_2(\Gamma)}$.

We can prove that (ii)$_2$ implies

(ii)$'_2$ $\exists C > 0$ such that $\forall f \in \ell_2(\Gamma)$ satisfies $\|\Delta_2 f\|_2 \geq C\|f\|_2$.

(ii)$'_2$ $\Leftrightarrow$ $\|\Delta_2\|_{\ell_2(\Gamma)\to\ell_2(\Gamma)} \geq C$, that is,
the spectrum of Δ_2 is bounded below by $C > 0$.

Summary and Future Issues

Summary

If Γ does not satisfy (ii) in the theorem, then Γ does not have Property $(F_{\ell^p(\Gamma)})$.

A graph property	$\Rightarrow$	A group property
(not (ii) in the theorem)		(without Property $(F_{\ell^p(\Gamma)})$)

Future Issues

Actually, it is not easy to make sure of the condition (ii) in the theorem. So we should find some example not satisfying (ii) in the theorem.

References

- M. Bourdon, F. Martin, A. Valette, *Vanishing and non-vanishing for the first L^p-cohomology of groups*, Comment. Math. Helv. **80** (2005), 377–389.
- M. Tanaka, *Property (T_B) and Property (F_B) restricted to an irreducible representation*, preprint.
- G. Yu, *Hyperbolic groups admit proper affine isometric actions on ℓ^p-spaces*, Geom. Funct. Anal. **15** (2005), 1144–1151.

Thank you very much.