Property (T) for uniformly convex uniformly smooth Banach spaces

Mamoru Tanaka (Mathematical Institute Tohoku University)

Żuk gives a sufficient condition for a finitely generated group to have Kazhdan's property (T) ([Żuk03]). We generalize a Żuk's result to the analogue of the notion for uniformly convex and uniformly smooth Banach spaces.

1 Preliminaries

Definition. A (real or complex) Banach space $(B, \|\ \|)$ is called uniformly convex if for every $\epsilon > 0$ there is a constant $\delta(\epsilon) > 0$ such that $\|u + v\| \leq 2 - \delta(\epsilon)$ for every $u, v \in B$ satisfying $\|u\| = \|v\| = 1$ and $\|u - v\| \geq \epsilon$. $(B, \|\ \|)$ is called uniformly smooth if the dual space $(B^*, \|\ \|_*)$ is uniformly convex. Hereafter uniformly convex and uniformly smooth is abbreviated to ucus.

If B is a ucus Banach space, there is a one-to-one duality map $^* : B \to B^*$, $u \mapsto u^*$ which satisfies $\|u^*\|_* = \|u\|$ and $u^*(u) = \|u\|^2$. In particular, B is reflexive.

Example. Lebesgue spaces $L^p([0,1])$ with $1 < p < \infty$ are ucus. But $L^1([0,1])$ and $L^\infty([0,1])$ are not ucus.

$(B, \|\ \|)$: a ucus Banach space
Γ : a finitely generated group
S : a finite generating subset of Γ with $S = S^{-1}$ and id $\notin S$
$\rho : \Gamma \to O(B)$: a linear isometric Γ-representation on B where $O(B)$ is the group of all invertible linear isometries on $(B, \|\ \|)$
A linear isometric Γ-representation $\rho : \Gamma \to O(B)$ descends to a linear isometric Γ-representation $\rho' : \Gamma \to O(B')$ on $B' = B/B^{\rho(\Gamma)}$ where $B^{\rho(\Gamma)}$ is the closed subspace of Γ-fixed vectors.

Definition ([BFGM07]). Γ is said to have property (T_B) if for every $\rho : \Gamma \to O(B)$ there exists a constant $K(S, \rho) > 0$ such that $\rho' : \Gamma \to O(B')$ satisfies

$$\max_{s \in S} \|\rho'(s)u - u\|' \geq K(S, \rho)\|u\|' \qquad (1)$$

for every $u \in B'$ where $\|\ \|'$ is the norm on B'.

Theorem ([BFGM07]). For $1 < p < \infty$, Γ has Kazhdan's property (T) if and only if Γ has property $(T_{L^p([0,1])})$.

$L(S)$: an oriented finite graph whose vertex set is S and edge set is $T := \{(s, s') \in S \times S | s^{-1}s' \in S\}$
We can take S so that $L(S)$ is connected. Then $n(s) := |\{s' \in S | (s, s') \in T\}| > 0$ for $s \in S$.
The spaces $M := \{f : S \to B\}$ and $M_* := \{g^* : S \to (B)^*\}$ which have respective norms defined by

$$\|f\|_M^2 := \sum_{s \in S} \|f(s)\|^2 \frac{n(s)}{|T|}; \|g^*\|_{M_*}^2 := \sum_{s \in S} \|g^*(s)\|_*^2 \frac{n(s)}{|T|}$$

for $f \in M$ and $g^* \in M_*$ are ucus Banach spaces. Let

$$\langle f, g^* \rangle := \sum_{s \in S} g^*(s)(f(s)) \frac{n(s)}{|T|}$$

for $f \in M$ and $g^* \in M_*$. Then the map $^* : M \to M_*$, $f \mapsto f^*$ defined by $f^*(s) := (f(s))^*$ for every $s \in S$ is a one-to-one duality map.

Definition. For $f \in M$, we define

$$(\Delta_B f)(s) := f(s) - \frac{1}{n(s)} \sum_{(s,s') \in T} f(s').$$

The map Δ_B is an analogue of a discrete Laplace operator. For $f \in M$ and $g^* \in M_*$ we define

$$(Pf)(s) := f(s) - \sum_{s' \in S} f(s') \frac{n(s')}{|T|},$$

$$(P^*g^*)(s) := g^*(s) - \sum_{s' \in S} g^*(s') \frac{n(s')}{|T|}.$$

The maps $P : M \to M$ and $P^* : M_* \to M_*$ are projections into the "orthogonal " complements of subspaces which consist of constant maps in M_* and M respectively.

2 Results

Theorem. B : a ucus Banach space
Γ : a group generated by a finite subset S as above
If $L(S)$ is connected and

$$\lambda_B(L(S)) := \inf_{f \in M; Pf \neq 0} \frac{|\langle \Delta_B(Pf), P^*f^* \rangle|}{|\langle Pf, P^*f^* \rangle|} > \frac{1}{2},$$

then Γ has property (T_B). Moreover, we can take a constant $2 - 1/\lambda_B(L(S))$ as $K(S, \rho)$.

Remark. If B is a Hilbert space, $\lambda_B(L(S))$ is the smallest non-zero eigenvalue of the discrete Laplace operator acting on $l^2(L(S), n)$. In this case, this theorem is the Żuk's original one, and he gives a group generated by S for which $\lambda_B(L(S))$ can be computed easily.

References

[BFGM07] U. Bader, A. Furman, T. Gelander and N. Monod, Property (T) and rigidity for actions on Banach spaces, Acta Math. 198 (2007), no.1, 57–105.

[Żuk03] A. Żuk, Property (T) and Kazhdan constants for discrete groups Geom. funct. anal. 13 (2003), 643–670.