Harmonic maps into Busemann nonpositive curvature spaces

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We are concerned with a harmonic map into a Busemann nonpositive curvature space from a finitely generated group acting on the space by isometries.

1 Preliminaries

Definition . A complete metric space (N, d) is called a Busemann nonpositive curvature space (say simply Busemann NPC space).
$\Leftrightarrow$ (1)(geodesic) $\forall p, q \in N$ has a shortest geodesic (an isometric embedding of a closed interval) joining them.
(2) $\forall c_i : [0, l_i] \to N$: shortest geodesics $(i = 1, 2)$ $d(t) := d(c_1(tl_1), c_2(tl_2))$ is convex for $t \in [0, 1]$, that is, $d(t) \le (1-t)d(0) + td(t)$ for $\forall t \in [0, 1]$.

Examples.
(i) CAT(0) spaces (Riemannian manifolds with non-positive curvature, trees, Hilbert spaces, etc.)
(ii) Strictly convex Banach spaces ($L^p(\Omega)$ and l^p for $1 < p < \infty$, etc.)

Γ : a finite generating group (fix in the following)
S : the generating set of Γ (fix in the following)
ρ : an isometric Γ-representation on (N, d)
$(\mathcal{M}, d_{\mathcal{M}})$: the metric space of ρ-equivariant maps $f : \Gamma \to (N, d)$ (i.e. $f(\gamma) = \rho(\gamma)f(e)$ for $\forall \gamma \in \Gamma$)
The energy at $f \in \mathcal{M}$ is defined by

$$E(f) := \sum_{\gamma \in S} m(e, \gamma) d(f(e), f(\gamma))^2$$

where e is the identity of Γ and $m(e, \gamma)$ is a positive weight satisfying some condition. The energy E is non negative, convex and continuous.

Definition . A ρ-equivariant map f is harmonic $\Leftrightarrow$ $E(f) = \inf_{g \in \mathcal{M}} E(g)$.

In particular, if $E(f) = 0$, then f is a constant map and $f(e)$ is a fixed point of ρ.

To investigate harmonic maps, we study the norm of the gradient of the energy at $f \in \mathcal{M}$ defined by

$$|\nabla_- E|(f) := \max\Big(\limsup_{g \to f,\ g \in \mathcal{M}} \frac{E(f) - E(g)}{d_{\mathcal{M}}(f, g)}, 0 \Big).$$

The important property is that if $|\nabla_- E|(f) = 0$ then f is harmonic.

Proposition. $\inf_{f \in \mathcal{M}} |\nabla_- E|(f) = 0$.

$\Omega \subset 2^{2^{\mathbb{N}}}$: the set of all non-principal ultrafilters on $\mathbb{N}$
For a sequence of metric spaces $\{(N_n, d_n, o_n)\}_{n \in \mathbb{N}}$ with base points and $\omega \in \Omega$ we can define an ultralimit $\omega\text{-}\lim_n (N_n, d_n, o_n)$. Similarly, we can define limit representation, limit map, etc. The ultralimit of a sequence of Busemann NPC spaces is a complete geodesic space, but may not be a Busemann NPC space.

2 Results

Theorem 1. (N, d) : a Busemann NPC space
ρ : an isometric Γ-representation on (N, d)
Take $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}$ satisfying $|\nabla_- E_n|(f_n) \to 0$ as $n \to \infty$. If there exists $C > 0$ such that

$$|\nabla_- E|(f_n)^2 \ge C E(f_n) > 0 \quad \text{for} \quad \forall n \in \mathbb{N},$$

then for $\forall \omega \in \Omega$ and $\forall x_0 \in \Gamma$ the limit map $\omega\text{-}\lim_n f_n : \Gamma \to \omega\text{-}\lim_n (N, d, f_n(x_0))$ is a (ω-$\lim_n \rho_n$)-equivariant constant map.

Note that the existence of such a sequence $\{f_n\}_{n=1}^{\infty}$ follows from Proposition.

Theorem 2.
$\mathcal{L}$: a family of Busemann NPC spaces such that for $\forall \{(N_n, d_n)\}_{n \in \mathbb{N}} \subset \mathcal{L}$, $\forall \{r_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_{>0}$ and $\forall \{o_n \in N_n\}_{n \in \mathbb{N}}$ there exists $\omega \in \Omega$ satisfying $\omega\text{-}\lim_n (N_n, r_n d_n, o_n) \in \mathcal{L}$.
Suppose that for any $(N, d) \in \mathcal{L}$ and any isometric Γ-representation ρ on (N, d) there exists a ρ-equivariant constant map into N. Then there exists a constant $C > 0$ such that

$$|\nabla_- E|(f)^2 \ge C E(f)$$

holds for all $(N, d) \in \mathcal{L}$, isometric Γ-representation ρ on (N, d) and ρ-equivariant map f into N. The constant $C = C(\Gamma, \mathcal{L})$ should be independent of (N, d), ρ and f.

The simple examples of $\mathcal{L}$ are the set of all Hilbert spaces and the set of all CAT(0) spaces. In the case, the converse of Theorem 2 is holds, and the constant C is called the Kazhdan constant of Γ. There exist more general examples of $\mathcal{L}$.