The existence of a global fixed point of an isometric action on a metric space

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Introduction

The main theme of this thesis is to prove the existence of a global fixed point of an isometric action of a finitely generated group (and a compactly generated group) on a metric space with a convex metric.

We say a topological group G has *Property* (FH) if, for any infinite dimensional real Hilbert space H, every continuous isometric action of G on H has a global fixed point. For a σ-compact locally compact group, it is known that Property (FH) is equivalent to Kazhdan's Property (T), which we shortly call Property (T). Property (T) is a condition introduced by Kazhdan [Kaz67] and was defined in terms of unitary representations of a topological group in question. Property (T) has played important roles in many different subjects including the structure of infinite groups, combinatorics, operator algebras, ergodic theory, smooth dynamics, random walks and so on (see [BHV08]). Its important property is that locally compact groups with Property (T) are compactly generated. In particular, discrete groups with Property (T) are finitely generated. The special linear groups $SL_n(\mathbb{R})$, $n \geq 3$, and the symplectic groups $Sp_{2n}(\mathbb{R})$, $n \geq 2$, are known to have Property (T). On the other hand, free Abelian groups $\mathbb{R}^n$, $\mathbb{Z}^n$ and free groups are compactly generated but do not have Property (T). Until recently, many examples of groups with Property (T) have been found. For example, Żuk [Żuk03] gave a criterion for a finitely generated group to have Property (T), which was stated by means of only finitely many relations in the presentation of the group. Using this criterion, he showed, for example, that there are many infinite hyperbolic groups with Property (T) in terms of "random groups".

Izeki and Nayatani [IN05] obtained a sufficient condition for an isometric action α of a discrete group Γ on a Hadamard space Y to have a global fixed point. Here by a Hadamard space we mean a complete metric space with a "nonpositive curvature". For instance, Hilbert spaces and simply connected complete Riemannian manifolds with nonpositive sectional curvature are Hadamard spaces. Their sufficient condition is described in terms of an energy functional E_α on the space of all α-equivariant maps from a countable Γ-space equipped with an admissible weight into Y. If E_α vanishes at an α-equivariant map f, then the image of f

is a global fixed point of α. To prove the existence of a global fixed point, they used the gradient flow associated with E_α, which was introduced by Jost [Jos98] and Mayer [May98]. The gradient flow decreases E_α in the most efficient way.

Subsequently, for a finitely generated group Γ and a family $\mathcal{L}$ of Hadamard spaces which is stable under scaling ultralimit, Izeki, Kondo, and Nayatani [IKN09] studied a fixed-point property defined as follows: The group Γ has this property if, for any $Y \in \mathcal{L}$, every isometric action of Γ on Y has a global fixed point. This property is a generalization of Property (FH). Indeed, the family of all real Hilbert spaces is stable under scaling ultralimit. Furthermore, Γ has Property (FH) if and only if every isometric action of Γ on any real Hilbert space has a global fixed point. They gave a necessary and sufficient condition for Γ to have the fixed-point property for $\mathcal{L}$ in terms of E_α.

On the other hand, Bader, Furman, Gelander, and Monod [BFGM07] introduced another generalization of Property (FH) and a generalization of Property (T) to a Banach space B. They say a topological group G has *Property* (F_B) if every continuous affine isometric action of G on B has a global fixed point. Note that it follows from Mazur-Ulam theorem that an isometric action on a real Banach space is affine (see [BL00]). They also define Property (T_B), which is a generalization of Property (T) to continuous linear isometric representations on B. According to a theorem by Guichardet, a σ-compact locally compact group with Property (F_B) has Property (T_B). However, there exists a finitely generated group which has Property (T_B) but does not have Property (F_B) for some B. Furthermore, for a σ-compact locally compact groups and $1 < p < \infty$, Property (T) is equivalent to Property $(T_{L^p([0,1])})$. On the contrary, Yu [Yu05] proved that an infinite hyperbolic group G, which may have Property (T), does not have Property $(F_{L^p(G)})$ if p is large enough. As these results show, in general, Property (F_B) and Property (T_B) are different.

In this thesis, first, we generalize results in [IN05] and [IKN09] to the case of global Busemann nonpositive curvature spaces (Definition 1.1.1), whose typical examples are Hadamard spaces and the Lebesgue spaces L^p with $1 < p < \infty$. Let G be a compactly generated group such that a compact generating subset K has a probability measure supported on K. When G is a finitely generated group, such a measure always exists and is called a weight. Let α be a continuous isometric action of G on a global Busemann nonpositive curvature space N. With α and $1 \leq p \leq \infty$, we associate a nonnegative function $F_{\alpha,p}$ (Definition 2.1.1) defined on N by making use of K. For a finitely generated group Γ, if we take Γ as a countable Γ-space, then the energy $E_\alpha(f)$ coincides with $(F_{\alpha,2}(f(e)))^2/2$, where f is an α-equivariant map and e is the identity element of Γ. Note that $F_{\alpha,p}$ vanishes at $x_0 \in N$ if and only if x_0 is a global fixed point of α.

To prove the existence of a global fixed point, we investigate the absolute gradient $|\nabla_- F_{\alpha,p}|(x)$ of $F_{\alpha,p}$ at each $x \in N$ (Definition 2.2.1). It is a key feature of $|\nabla_- F_{\alpha,p}|$ that it gives the maximum descent of $F_{\alpha,p}$ around each point. In particular, if $|\nabla_- F_{\alpha,p}|$ vanishes at $x_0 \in N$, then x_0 minimizes $F_{\alpha,p}$ (Corollary 2.2.3). We will prove that there always exists a sequence of points such that $|\nabla_- F_{\alpha,p}|$ approaches zero (Lemma 2.2.5). Furthermore, without using Jost-Mayer's gradient flow, we will obtain the following theorem.

Theorem 1. *Let α be an isometric action of a compactly generated group G on a global Busemann nonpositive curvature space N, and $1 \leq p \leq \infty$. If there exists $C > 0$ such that $|\nabla_- F_{\alpha,p}|(x) \geq C$ for all $x \in N$ with $F_{\alpha,p}(x) > 0$, then α has a global fixed point.*

Theorem 1 generalizes a result in [IN05]. Indeed, to prove the existence of a global fixed point, they assume the existence of $C > 0$ satisfying the inequality $|\nabla_- E_\alpha|(f)^2 \geq C E_\alpha(f)$ for every α-equivariant map f. If we take Γ as a countable Γ-space, then the space of all α-equivariant maps can be identified with the Hadamard space Y on which Γ is acting, and this inequality is equivalent to the inequality $|\nabla_- F_{\alpha,2}|(x) \geq \sqrt{C/2}$ for all $x \in Y$ with $F_{\alpha,2}(x) > 0$ (see Section 3.1). However, our proof is totally different from the proof in [IN05].

Theorem 2. *Let Γ be a finitely generated group. Fix a finite generating subset K of Γ and a weight on K. Let $\mathcal{L}$ be a family of global Busemann nonpositive curvature spaces, and $1 \leq p \leq \infty$. Suppose that $\mathcal{L}$ is stable under scaling ultralimit. Then the following are equivalent*:

(i) *For any $N \in \mathcal{L}$, every isometric action of Γ on N has a global fixed point.*

(ii) *For any $N \in \mathcal{L}$ and isometric action α of Γ on N, there exists $C > 0$ such that $|\nabla_- F_{\alpha,p}|(x) \geq C$ for all $x \in N$ with $F_{\alpha,p}(x) > 0$.*

Furthermore, in (ii)*, C can be a constant independent of N and α.*

Theorem 2 is a generalization of a result in [IKN09] and an improvement of a result in [Tan]. An example of such a family $\mathcal{L}$ is the family of all L^p, where p is a fixed number with $1 < p < \infty$. Other examples will be given in Section 3.2.

Second, for an affine isometric action α of a finitely generated group Γ on a Banach space B, we study the existence of a global fixed point. In this thesis, we will introduce a new property defined in terms of $|\nabla_- F_{\alpha,p}|$, and will investigate relations between Property (F_B), Property (T_B) and the new one. To describe the new property, we briefly recall the definition of affine isometric actions. An isometric action α of Γ on B is said to be *affine* if it is written as $\alpha(\gamma)v =$

$\pi(\gamma)v+c(\gamma)$ for each $v \in B$ and $\gamma \in \Gamma$, where π is a linear isometric representation of Γ on B and c is a π-cocycle. (We will define them in Section 2.3 in detail.) We call π the *linear part* of α. An affine isometric action α of Γ on B with a linear part π descends to an affine isometric action α' of Γ on $B' := B/B^{\pi(\Gamma)}$, where $B^{\pi(\Gamma)}$ is the closed subspace of all invariant vectors of π. Let $1 < p < \infty$. We say Γ has *Property* $(AG_{B,p})$, if, for any affine isometric action α of Γ on B, there exists $C > 0$ such that $|\nabla_- F_{\alpha',p}|(v') \geq C$ for all $v' \in B'$ with $F_{\alpha',p}(v') > 0$. It is known that, for an infinite dimensional Hilbert space H, Property $(AG_{H,2})$ is equivalent to Property (FH). In the case of Banach spaces, we will obtain the following theorem.

Theorem 3. *If Γ has Property (F_B), then it has Property $(AG_{B,p})$ for all $1 < p < \infty$.*

It should be remarked that the converse of Theorem 3 is false. Indeed, $\mathbb{Z}$ has Property $(AG_{\mathbb{R},p})$ but does not satisfy Property $(F_{\mathbb{R}})$ (see Section 4.3). However, we will obtain Theorem 4 below. Note that, for a linear isometric representation π of Γ on B having no non-trivial invariant vector, the first cohomology $H^1(\Gamma, \pi)$ vanishes if and only if every affine isometric action with linear part π has a global fixed point.

Theorem 4. *Let π be a linear isometric representation of Γ on B, and $1 \leq p \leq \infty$. Suppose that B is strictly convex and π has no non-trivial invariant vector. Then the following are equivalent*:

(i) *The first cohomology $H^1(\Gamma, \pi)$ vanishes.*

(ii) *For any affine isometric action α of Γ on B with the linear part π, there exists $C > 0$ such that $|\nabla_- F_{\alpha,p}|(v) \geq C$ for all $v \in B$ with $F_{\alpha,p}(v) > 0$.*

Furthermore, in (ii), *C can be a constant independent of α.*

Although strictly convex Banach spaces are global Busemann nonpositive curvature spaces, this theorem is not a corollary of Theorem 2, since the family consisting of one separable Banach space is not stable under scaling ultralimit (see Section 1.2). Also, we can not use Theorem 4 to prove Theorem 3, because some affine isometric action on B' may not extend to B.

For an affine isometric action α of Γ on B, $|\nabla_- F_{\alpha,p}|$ with $1 < p < \infty$ can be written explicitly in the following case. We choose a finite generating subset K of Γ and a weight m on K to define $F_{\alpha,p}$. Suppose that B is either strictly convex, smooth and real, or uniformly convex and uniformly smooth. Besides, suppose that K is symmetric and m is symmetric. Let $j(u)$ denote the support functional of a non-trivial $u \in B$, that is, the continuous linear functional on B satisfying $j(u)u = \|u\|$ and $\|j(u)\|_{B^*} = 1$. For the trivial vector $0 \in B$, we set $j(0)$ to be the zero functional on B. Then $|\nabla_- F_{\alpha,p}|(v)$ is expressed as

$$\frac{2}{F_{\alpha,p}(v)^{p-1}} \left\| \sum_{\gamma \in K} \|v - \alpha(\gamma)v\|^{p-1} m(\gamma) \operatorname{Re} j(v - \alpha(\gamma)v) \right\|_{B^*}$$

for all $v \in B$ with $F_{\alpha,p}(v) > 0$. Here, for $g \in B^*$, $\operatorname{Re} g$ denotes the real-valued part of g. If B is real, Re is trivial. In particular, if B is $L^p(W,\nu)$, where (W,ν) is a measure space, then $|\nabla_- F_{\alpha,p}|(f)$ is expressed as $2\|G\|_{L^q(\nu)}/F_{\alpha,p}(f)^{p-1}$ for $f \in L^p(W,\nu)$ with $F_{\alpha,p}(f) > 0$. Here q is the conjugate exponent of p, that is, $q = p/(p-1)$, and

$$G(x) = \sum_{\gamma \in K} |f(x) - \alpha(\gamma)f(x)|^{p-2} \operatorname{Re}(f(x) - \alpha(\gamma)f(x)) m(\gamma)$$

for all $x \in W$, where $\operatorname{Re}(a)$ denotes the real part of a complex number a, and $|f(x) - \alpha(\gamma)f(x)|^{p-2}$ is defined to be zero if $f(x) = \alpha(\gamma)f(x)$ and $p < 2$.

By making use of explicit expression of $|\nabla_- F_{\alpha,p}|$, we will obtain the following corollary of Theorem 4.

Corollary. *Let π be a linear isometric representation of Γ on B and $1 < p < \infty$. Suppose that B is either strictly convex, smooth and real, or uniformly convex and uniformly smooth, and π has no non-trivial invariant vector. Then $H^1(\Gamma, \pi)$ vanishes if and only if there exists $C > 0$ such that*

$$\left\| \sum_{\gamma \in K} m(\gamma) \|c(\gamma)\|^{p-1} \operatorname{Re} j(c(\gamma)) \right\|_{B^*} \geq C \|c\|_p^{p-1}$$

for all π-cocycles c.

In Section 4.2, for a finitely generated group Γ, we will apply this corollary to the left regular representation $\lambda_{\Gamma,p}$ of Γ on $\ell^p(\Gamma)$ with $1 < p < \infty$. Then, using the p-Laplacian on the Cayley graph of Γ, we will state a necessary and sufficient condition for the vanishing of $H^1(\Gamma, \lambda_{\Gamma,p})$.

On the other hand, one may expect that there is also some relation between Property (T_B) and Property $(AG_{B,p})$. We can show the following proposition.

Proposition 5. *Suppose that Γ is Abelian, K is symmetric, m is symmetric, and B is uniformly convex, uniformly smooth and real. If Γ has Property (T_B), then it has Property $(AG_{B,p})$ for all $1 < p < \infty$.*

Finally, we will generalize a result in [Żuk03] mentioned above to the case of uniformly convex and uniformly smooth real Banach spaces. Let Γ be a finitely generated group. Given a symmetric finite generating subset K of Γ, not containing e, we will construct a connected finite oriented graph $L(K)$ which is the same graph as the one constructed in [Żuk03]. For a uniformly convex and uniformly smooth real Banach space B and $1 < p < \infty$, we will introduce an invariant $\lambda_{B,p}(L(K))$ (Definition 5.2.3). Then we will prove the following

Theorem 6. *If $\lambda_{B/\tilde{B},p}(L(K)) > 1/2$ for every closed subspace $\tilde{B}$ of B, then Γ has Property (T_B).*

Note that $\lambda_{B/\tilde{B},p}(L(K))$ is independent of the linear isometric representations of Γ. For a Hilbert space H, $\lambda_{H,2}(L(K))$ is the smallest non-zero eigenvalue of the discrete Laplacian acting on $l^2(L(K), \mathrm{deg})$. In this case, our theorem coincides with Żuk's original one, and there exists a finitely generated group such that $\lambda_{H,2}(L(K))$ can be computed easily. However, in the case of Banach spaces, the computation of $\lambda_{B/\tilde{B},2}(L(K))$ is not easy. Indeed, although B is a Lebesgue space, $B/\tilde{B}$ may not be a Lebesgue space.

This thesis is organized as follows. In Chapter 1, we review relevant definitions and properties of global Busemann nonpositive curvature spaces, the ultralimit of a sequence of metric spaces, strictly convex and smooth Banach spaces, and uniformly convex and uniformly smooth Banach spaces. In Chapter 2, we define $F_{\alpha,p}$ and $|\nabla_- F_{\alpha,p}|$, and describe their basic properties. Moreover, we review the definition of the first cohomology, and give an explicit expression of $|\nabla_- F_{\alpha,p}|$ for an affine isometric action α. In Chapter 3, we prove Theorem 1 and Theorem 2, and give examples of families satisfying the assumption of Theorem 2. In Chapter 4, we review the definitions of properties (F_B), (T_B) and $(AG_{B,p})$, and prove Theorem 3, Theorem 4 and Proposition 5. In Chapter 5, we generalize a result of Żuk, that is, prove Theorem 6.

Chapter 1

Metric spaces

In this chapter, we briefly review main geometric objects studied in this thesis, namely, global Busemann nonpositive curvature spaces, the ultralimit of a sequence of metric spaces, strictly convex and smooth Banach spaces, and uniformly convex and uniformly smooth Banach spaces. We also recall some basic facts which will be used throughout the thesis.

1.1 Global Busemann NPC spaces

We first recall the definition and several properties of global Busemann nonpositive curvature spaces. The notion of a global Busemann nonpositive curvature space was introduced by Busemann [Bus55]. Most of results in this section can be found in [Jos97] and [Pap05].

Given a pair of points x and y in a metric space (M, d), a *shortest geodesic* joining x to y is a map $c : [0, l] \to M$ which satisfies $c(0) = x$, $c(l) = y$ and $d(c(t), c(s)) = |t - s|$ for all $t, s \in [0, l]$. Note that $l = d(x, y)$. A metric space is called a *geodesic space* if every pair of points has a shortest geodesic joining them.

Definition 1.1.1. A complete geodesic space (N, d) is called a *global Busemann nonpositive curvature space* (or shortly a *global Busemann NPC space*) if the *Busemann NPC inequality*

$$d\left(c_1\left(\frac{l_1}{2}\right), c_2\left(\frac{l_2}{2}\right)\right) \leq \frac{1}{2} d(c_1(l_1), c_2(l_2))$$

holds for every pair of shortest geodesics $c_i : [0, l_i] \to N$ $(i = 1, 2)$ satisfying $c_1(0) = c_2(0)$.

Example 1.1.2. Hadamard spaces are global Busemann NPC spaces. In particular, simply connected complete Riemannian manifolds with nonpositive sectional curvature, trees, Bruhat-Tits buildings, and Hilbert spaces are global Busemann NPC spaces. Strictly convex Banach spaces, which will be defined in Section 1.3, are also global Busemann NPC spaces.

A Lebesgue space L^p with $p \neq 2$ is not a Hadamard space, but L^p with $1 < p < \infty$ is a global Busemann NPC space. A difference between Hadamard spaces and global Busemann NPC spaces may be typically seen in that between Hilbert spaces and strictly convex Banach spaces.

For a global Busemann NPC space (N, d), the metric is convex in the following sense.

Theorem 1.1.3 (cf. [Jos97]). *Let $c_0 : [0, l_0] \to N$ and $c_1 : [0, l_1] \to N$ be shortest geodesics in (N, d). Then the function $d(c_0(l_0 t), c_1(l_1 t))$ of t is convex on $[0, 1]$, that is,*

$$d(c_0(tl_0), c_1(tl_1)) \leq (1 - t)d(c_0(0), c_1(0)) + td(c_0(l_0), c_1(l_1))$$

for all $t \in [0, 1]$.

The convexity of the metric is the most important feature of global Busemann NPC spaces.

It follows from this theorem that each pair of points in a global Busemann NPC space is joined by precisely one shortest geodesic. Moreover, a global Busemann NPC space (N, d) is contractible. Indeed, fix an arbitrary $x_0 \in X$, and for each $x \in N$ set $c_x : [0, l_x] \to N$ to be the shortest geodesic joining x_0 to x. Then $F : N \times [0, 1] \to N$ defined by $F(x, t) := c_x(tl_x)$ for $x \in N$ and $t \in [0, 1]$ is continuous in x and t, because

$$\begin{aligned} d(F(x,t), F(y,s)) &= d(c_x(tl_x), c_y(sl_y)) \\ &\leq d(c_x(tl_x), c_x(sl_x)) + d(c_x(sl_x), c_y(sl_y)) \\ &\leq |t - s| l_x + sd(x, y) \end{aligned}$$

for all $x, y \in N$ and $t, s \in [0, 1]$ by Theorem 1.1.3. Therefore the identity map on N and the constant map x_0 are homotopic.

1.2 Ultralimit of metric spaces

Next, we recall the definition and several properties of the ultralimit of a sequence of metric spaces. Basic references of this section are [BH99] and [Kap01]. We denote by $\mathbb{N}$ the set of all natural numbers, and by $\mathbb{R}$ the set of all real numbers.

Definition 1.2.1. A *filter* ω on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ satisfying the following:

(i) $\emptyset \notin \omega$ and $\mathbb{N} \in \omega$, where $\emptyset$ is the empty set.

(ii) If $B \in \omega$ and $B \subset A$, then $A \in \omega$.

(iii) If $A, B \in \omega$, then $A \cap B \in \omega$.

In addition, if a filter ω satisfies that

(iv) for each $A \in \mathbb{N}$, $A \in \omega$ or $A^c \in \omega$,

then ω is called an *ultrafilter*, where A^c is the complement of A. On the other hand, if a filter ω satisfies that

(v) no finite subset of $\mathbb{N}$ is not in ω,

then ω is said to be *non-principal*.

An example of a non-principal ultrafilter on $\mathbb{N}$ can be given as follows: We can easily see that $\omega' := \{A^c \subset \mathbb{N} : |A| < \infty\}$ is a non-principal filter, where $|A|$ is the cardinal number of A. Considering all non-principal filters containing ω', we obtain a maximal one ω_0 by Zorn's Lemma. This ω_0 must be a non-principal ultrafilter. Suppose not. Then there exists $A \subset \mathbb{N}$ such that $A \notin \omega_0$ and $A^c \notin \omega_0$. Then

$$\omega_1 := \omega_0 \cup \{C \subset \mathbb{N} : A \cap B \subset C \text{ for some } B \in \omega_0\}$$

is also a non-principal filter. Indeed, we have the following: (i) $\mathbb{N} \in \omega_1$. If $\emptyset \in \omega_1$, then there exists $B \in \omega_0$ such that $A \cap B = \emptyset$, that is, $B \subset A^c$. Thus $A^c \in \omega_0$. This is a contradiction, that is, $\emptyset \notin \omega_1$. (ii) If $C \in \omega_1$ and $C \subset D$, then there exists $B \in \omega_0$ such that $A \cap B \subset D$. Hence $D \in \omega_1$. (iii) Since, for $C_1, C_2 \in \omega_1$, there exists $B_1, B_2 \in \omega_0$ such that $A \cap B_1 \subset C_1$ and $A \cap B_2 \subset C_2$, we have $A \cap B_1 \cap B_2 \subset C_1 \cap C_2$. Because $A \cap B_1 \cap B_2 \in \omega_1$, $C_1 \cap C_2 \in \omega_1$. (v) If there exists $E \in \omega_1$ such that $|E| < \infty$, then there exists $B \in \omega_0$ such that $|A \cap B| < \infty$. Since $(A \cap B)^c \in \omega_0$, we get $B \cap (A \cap B)^c = B \cap A^c \in \omega_0$. Thus $A^c \in \omega_0$. This is a contradiction, that is, no finite subset is in ω_1. Therefore ω_1 is a non-principal filter. Since $A \in \omega_1$, ω_0 is properly contained in ω_1. This contradicts the assumption that ω_0 is maximal. Therefore ω_0 is a non-principal ultrafilter.

Throughout this section, let ω be a non-principal ultrafilter on $\mathbb{N}$.

For a bounded sequence $\{a_n\} \subset \mathbb{R}$, there exists a unique real number l such that $\{n : |a_n - l| < \epsilon\} \in \omega$ for all $\epsilon > 0$ (see [BH99, Lemma 5.49]). We denote this

l by a_ω or ω-$\lim_n a_n$, and call it the *ultralimit* of $\{a_n\}$. If a sequence $\{a_n\} \subset \mathbb{R}$ converges to a_0 in the usual sense, then $a_\omega = a_0$. Thus we can regard the ultralimit of a bounded sequence of real numbers as a limit of the sequence in some sense.

Lemma 1.2.2. *Ultralimit ω-lim is linear, that is, ω-$\lim_n(a_n + b_n) = a_\omega + b_\omega$ and ω-$\lim_n(ra_n) = ra_\omega$ for all bounded sequences $\{a_n\}, \{b_n\} \subset \mathbb{R}$ and $r \in \mathbb{R}$.*

Proof. For $\epsilon > 0$, we have

$$\{n : |a_n - a_\omega| < \epsilon/2, |b_n - b_\omega| < \epsilon/2\} \subset \{n : |(a_n + b_n) - (a_\omega + b_\omega)| < \epsilon\}.$$

Using (ii) and (iii) in Definition 1.2.1, we obtain ω-$\lim_n(a_n+b_n) = a_\omega+b_\omega$. On the other hand, if $r \geq 0$, then $\{n : |ra_n - ra_\omega| < \epsilon\} = \{n : |a_n - a_\omega| < \epsilon/|r|\} \in \omega$ for $\epsilon > 0$. Hence we get ω-$\lim_n(ra_n) = ra_\omega$. If $r = 0$, then obviously ω-$\lim_n(ra_n) = ra_\omega$. □

Lemma 1.2.3. *If a pair of bounded sequences $\{a_n\}, \{b_n\} \subset \mathbb{R}$ satisfies $a_n \leq b_n$ for each $n \in \mathbb{N}$, then $a_\omega \leq b_\omega$.*

Proof. If $n \in \mathbb{N}$ satisfies $|a_n - a_\omega| < \epsilon$ for $\epsilon > 0$, then $a_n \in (a_\omega - \epsilon, \infty)$. From the assumption, $b_n \in (a_\omega - \epsilon, \infty)$. Thus

$$\{n : |a_n - a_\omega| < \epsilon, |b_n - b_\omega| < \epsilon\} \subset \{n : b_n \in (a_\omega - \epsilon, \infty) \cap (b_\omega - \epsilon, b_\omega + \epsilon)\}.$$

It follows from (i), (ii) and (iii) in Definition 1.2.1 that $(a_\omega-\epsilon, \infty)\cap(b_\omega-\epsilon, b_\omega+\epsilon) \neq \emptyset$ for all $\epsilon > 0$. Therefore we have $a_\omega \leq b_\omega$. □

Lemma 1.2.4. *For any $r > 0$, every bounded sequence $\{a_n\}$ such that $a_n \geq 0$ for all $n \in \mathbb{N}$ satisfies $a_\omega^r = \omega$-$\lim_n(a_n^r)$.*

Proof. Since $a_n \geq 0$, we have $a_\omega \geq 0$ by the definition of the ultralimit. If $a_\omega = 0$, then $\{n : a_n^r < \epsilon\} = \{n : a_n < \epsilon^{1/r}\} \in \omega$ for any $\epsilon > 0$. Hence ω-$\lim_n(a_n^r) = 0$. Suppose $a_\omega > 0$. If $r = 1$, then the lemma is obvious. In the case that $0 < r < 1$, we use an inequality in [HLP52, (2.15.2)]: $a^r - b^r \leq rb^{r-1}(a - b)$ for $a, b > 0$. Let $0 < \epsilon < a_\omega/2$. If n satisfies $|a_n - a_\omega| < \epsilon$, then we have

$$a_n^r - a_\omega^r \leq ra_\omega^{r-1}(a_n - a_\omega) < r\left(\frac{a_\omega}{2}\right)^{r-1}\epsilon$$

and

$$a_\omega^r - a_n^r \leq ra_n^{r-1}(a_\omega - a_n) < r(a_\omega - \epsilon)^{r-1}\epsilon < r\left(\frac{a_\omega}{2}\right)^{r-1}\epsilon.$$

These inequalities imply

$$\{n : |a_n - a_\omega| < \epsilon\} \subset \left\{n : |a_n^r - a_\omega^r| < r\left(\frac{a_\omega}{2}\right)^{r-1}\epsilon\right\}.$$

Using (ii) in Definition 1.2.1, $\{n : |a_n^r - a_\omega^r| < r(a_\omega/2)^{r-1}\epsilon\} \in \omega$ for all $\epsilon > 0$. Hence $a_\omega^r = \omega\text{-}\lim_n(a_n^r)$. In the case that $r > 1$, we use an inequality in [HLP52, (2.15.1)]: $a^r - b^r \leq ra^{r-1}(a-b)$ for $a, b > 0$. Let $0 < \epsilon < a_\omega$. If n satisfies $|a_n - a_\omega| < \epsilon$, then we obtain

$$a_n^r - a_\omega^r \leq ra_n^{r-1}(a_n - a_\omega) < r(a_\omega + \epsilon)^{r-1}\epsilon < r(2a_\omega)^{r-1}\epsilon$$

and

$$a_\omega^r - a_n^r \leq ra_\omega^{r-1}(a_\omega - a_n) < r(2a_\omega)^{r-1}\epsilon.$$

As the case that $0 < r < 1$, these inequalities imply $a_\omega^r = \omega\text{-}\lim_n(a_n^r)$. □

Lemma 1.2.5. *Let S be a finite set. For each $i \in S$, take an arbitrary bounded sequence $\{a_n^i\} \subset \mathbb{R}$. Then $\max_{i\in S} a_\omega^i = \omega\text{-}\lim_n(\max_{i\in S} a_n^i)$.*

Proof. Set b_n to be $\max_{i\in S} a_n^i$ for each n. Due to Lemma 1.2.3, we have $a_\omega^i \leq b_\omega$ for all $i \in S$. Hence $\max_{i\in S} a_\omega^i \leq b_\omega$. We must prove $\max_{i\in S} a_\omega^i \geq b_\omega$. Suppose not. Take $i_0 \in S$ satisfying $a_\omega^{i_0} = \max_{i\in S} a_\omega^i$, and set $\epsilon = (b_\omega - a_\omega^{i_0})/2$. If $\{n : |b_\omega - a_n^i| < \epsilon\} \in \omega$ for $i \in S$, then we have $a_\omega^{i_0} < b_\omega - \epsilon \leq a_\omega^i$. This contradicts the assumption that $a_\omega^{i_0}$ is maximal. Hence $\{n : |b_\omega - a_n^i| \geq \epsilon\} \in \omega$ for all $i \in S$. By (iii) in Definition 1.2.1, $\cap_{i\in S}\{n : |b_\omega - a_n^i| \geq \epsilon\} \in \omega$. Note that if $A \in \omega$, then $A^c \notin \omega$ from (i) and (iii) in Definition 1.2.1. Hence, by (iv) in Definition 1.2.1, $\cup_{i\in S}\{n : |b_\omega - a_n^i| < \epsilon\} \notin \omega$. However, $\{n : |b_\omega - b_n| < \epsilon\} \subset \cup_{i\in S}\{n : |b_\omega - a_n^i| < \epsilon\}$, and the right hand side is in ω. This contradicts (ii) in Definition 1.2.1. □

A pair of a metric space and a point in the metric space is called a *metric space with a base point*.

Definition 1.2.6. Let $\{(M_n, d_n, o_n)\}$ be a sequence of metric spaces with base points. We denote by M_∞ the set of all $(x_n) \in \prod_{n\in\mathbb{N}} M_n$ such that $\{d_n(x_n, o_n)\}$ is bounded independently of n. We say that $(x_n) \in M_\infty$ and $(y_n) \in M_\infty$ are *equivalent* if $\omega\text{-}\lim_n d_n(x_n, y_n) = 0$. We denote by x_ω or $\omega\text{-}\lim_n x_n$ the equivalence class of (x_n), and by M_ω the set of all equivalence classes. We endow M_ω with the metric $d_\omega(x_\omega, y_\omega) := \omega\text{-}\lim_n d_n(x_n, y_n)$ for each $x_\omega, y_\omega \in M_\omega$, where (x_n) and (y_n) are representatives of x_ω and y_ω respectively. The metric space (M_ω, d_ω) is called the *ultralimit* of $\{(M_n, d_n, o_n)\}$ with respect to ω, and also written as $\omega\text{-}\lim_n(M_n, d_n, o_n)$.

Note that the ultralimit of a sequence of metric spaces is complete (see [BH99], Lemma 5.53]).

Let (M, d, o) be a proper metric space with a base point. Here proper means that any bounded closed set is compact. Set $(M_n, d_n, o_n) = (M, d, o)$ for all n. Then (M_ω, d_ω) and (M, d) are isometric (see [BH99, Remark 5.55]).

The ultralimit of a sequence of Banach spaces with base points becomes a Banach space (see [Hei80]), and the ultralimit is independent of the base points. Hence, if a Banach space $(B, \| \ \|)$ is finite dimensional, then the ultralimit of $\{(B_n, \| \ \|_n, o_n)\}$ such that $(B_n, \| \ \|_n) = (B, \| \ \|)$ for all n is isometrically isomorphic to $(B, \| \ \|)$. However, if $(B, \| \ \|)$ is infinite dimensional, then $\omega\text{-}\lim_n(B_n, \| \ \|_n, o_n)$ may not be isometrically isomorphic to $(B, \| \ \|)$. Indeed, suppose $(B, \| \ \|)$ is separable, and o_n is the origin of B for each n. Then $\omega\text{-}\lim_n(B_n, \| \ \|_n, o_n)$ is not separable, because of the following: We can take $\{v_n\} \subset B$ such that $\|v_n - v_m\| > 1/2$ and $\|v_n\| = 1$ for all n and m with $n \neq m$. For $l \in \mathbb{R}$, we define $[l]$ as $n \in \mathbb{N}$ with $n \leq l < n+1$. If $b > a > 0$, then there exists $n_0 \in \mathbb{N}$ such that $[k^a] < [k^b]$ for all $k \geq n_0$. Hence $\omega\text{-}\lim_k \|v_{[k^a]} - v_{[k^b]}\| \geq 1/2$ and $\omega\text{-}\lim_k \|v_{[k^a]}\| = \omega\text{-}\lim_k \|v_{[k^b]}\| = 1$. Thus $|\{\omega\text{-}\lim_k v_{[k^a]} \in B_\omega : a > 0\}|$ is continuous, and the open balls $B(\omega\text{-}\lim_k v_{[k^a]}, 1/4)$ are mutually disjoint. Therefore $\omega\text{-}\lim_n(B_n, \| \ \|_n, o_n)$ is not separable.

The ultralimit of a sequence of geodesic spaces is a geodesic space (see [BH99, Exercise 5.54]). However, the ultralimit of a sequence of global Busemann NPC spaces may not be a global Busemann NPC space. For example, $(\mathbb{R}^2, \| \ \|_p)$ with $1 < p < \infty$ is a global Busemann NPC space, where $\| \ \|_p$ is defined by $\|u\|_p := (|u^1|^p + |u^2|^p)^{1/p}$ for each $u = (u^1, u^2) \in \mathbb{R}^2$. However, $(\mathbb{R}^2, \| \ \|_\omega) = \omega\text{-}\lim_n(\mathbb{R}^2, \| \ \|_n, o)$ is not a global Busemann NPC space, where o is the origin of $\mathbb{R}^2$. Indeed, $u = (1, 0)$, $v = (2, 0)$, and $w = (0, 1)$ in $\mathbb{R}^2$ satisfy

$$\begin{aligned}
\|u - v\|_n &= (|u^1 - v^1|^n + |u^2 - v^2|^n)^{1/n} = 1, \\
\|u - w\|_n &= (|u^1 - w^1|^n + |u^2 - w^2|^n)^{1/n} = 2^{1/n}, \\
\|v - w\|_n &= (|v^1 - w^1|^n + |v^2 - w^2|^n)^{1/n} = (2^n + 1)^{1/n}, \\
\left\| u - \frac{v + w}{2} \right\|_n &= \left(\left| u^1 - \frac{v^1 + w^1}{2} \right|^n + \left| u^2 - \frac{v^2 + w^2}{2} \right|^n \right)^{1/n} = \frac{1}{2}
\end{aligned}$$

for all $n \in \mathbb{N}$. Set $(u_n), (v_n), (w_n) \in \prod_{n \in \mathbb{N}} \mathbb{R}^2$ to be $u_n = u$, $v_n = v$, and $w_n = w$ for all n. Then we have $\|u_\omega - v_\omega\|_\omega = 1$, $\|u_\omega - w_\omega\|_\omega = 1$, and $\|v_\omega - w_\omega\|_\omega = 2$. On the other hand, $u_\omega \neq (v_\omega + w_\omega)/2$. Hence there exist two different shortest geodesics joining v_ω to w_ω. The one is the line segment from v_ω to w_ω, because the line segment from u to w is a shortest geodesic joining them. The other is the path from v_ω through u_ω to w_ω consisting of two line segments joining v_ω to u_ω and u_ω to w_ω, because $\|u_\omega - v_\omega\|_\omega + \|u_\omega - w_\omega\|_\omega = \|v_\omega - w_\omega\|_\omega$. Therefore $(\mathbb{R}^2, \| \ \|_\omega)$ is not a global Busemann NPC space.

Definition 1.2.7. A family $\mathcal{L}$ of metric spaces is said to be *stable under scaling ultralimit* if, for any $\{(M_n, d_n, o_n)\} \subset \mathcal{L}$, $\{r_n\} \subset \mathbb{R}$ with $r_n > 0$ for all n, and non-principal ultrafilter ω on $\mathbb{N}$, the ultralimit $\omega\text{-}\lim_n(M_n, r_n d_n, o_n)$ is in $\mathcal{L}$.

For example, for a fixed p with $1 \leq p < \infty$, the family of all L^p is stable under scaling ultralimit (see [AK90] and [Hei80]). In particular, the family of all Hilbert spaces is stable under scaling ultralimit.

1.3 Strictly convex Banach spaces

Next, we review the definitions and several properties of strictly convex Banach spaces, smooth Banach spaces, uniformly convex Banach spaces and uniformly smooth Banach spaces. Basic references are [BL00], [LT77] and [LT79]. We denote by $(B^*, \| \|_{B^*})$ the dual Banach space of a Banach space $(B, \| \|)$.

Definition 1.3.1. A Banach space $(B, \| \|)$ is said to be *strictly convex* if $\|v+u\| < 2$ for all $v, u \in B$ with $v \neq u$, $\|v\| \leq 1$ and $\|u\| \leq 1$.

Strictly convex Banach spaces are global Busemann NPC spaces as mentioned in Section 1.1.

A support functional at v in a Banach space $(B, \| \|)$ is $f \in B^*$ satisfying $\|f\|_{B^*} = 1$ and $f(v) = \|v\|$.

Definition 1.3.2. A Banach space is said to be *smooth* if every non-trivial vector has a unique support functional.

We denote by $j(v)$ the support functional at a non-trivial vector v in a smooth Banach space B, and call j the *duality map*. For the trivial vector 0 of B, we set $j(0)$ to be the zero functional on B. If B is real, then

$$j(v)u = \lim_{t \to 0} \frac{\|v + tu\| - \|v\|}{t}$$

for all $v \in B \backslash \{0\}$ and $u \in B$. Furthermore, if B^* is strictly convex (resp. smooth), then B is smooth (resp. strictly convex).

Definition 1.3.3. A Banach space $(B, \| \|)$ is said to be *uniformly convex* if the *modulus of convexity* of B

$$\delta_B(\epsilon) := \inf \left\{ 1 - \frac{\|u + v\|}{2} : \|u\| \leq 1, \|v\| \leq 1 \text{ and } \|u - v\| \geq \epsilon \right\}$$

is positive for all $\epsilon > 0$.

A uniformly convex Banach space is strictly convex obviously. The closed subspaces and the quotient spaces of a uniformly convex Banach space are also uniformly convex. A uniformly convex real Banach space B is reflexive, that is, $(B^*)^* = B$.

Definition 1.3.4. A Banach space $(B, \| \ \|)$ is said to be *uniformly smooth* if the *modulus of smoothness* of B

$$\rho_B(\tau) := \sup \left\{ \frac{\|u+v\|}{2} + \frac{\|u-v\|}{2} - 1 : \|u\| \leq 1 \text{ and } \|v\| \leq \tau \right\}$$

satisfies that $\rho_B(\tau)/\tau \to 0$ whenever $\tau \searrow 0$.

A uniformly smooth Banach space B is smooth. Indeed, if B is not smooth, then there exist $v \in B$ and $f, g \in B^*$ satisfying $f \neq g$, $\|f\|_{B^*} = \|g\|_{B^*} = 1$ and $f(v) = g(v) = \|v\| = 1$. Since $f \neq g$, there exists a non-trivial vector w such that $f(w) \neq g(w)$. Set $w_1 = w - g(w)v$. Then $f(w_1) \neq 0$ and $g(w_1) = 0$. Set $w_2 = \overline{f(w_1)} w_1$, where $\overline{f(w_1)}$ is the conjugate of the complex number $f(w_1)$. Then $f(w_2)$ is real. Retaking w_2 as $-w_2$ if necessary, we may assume that $f(w_2) > 0$. Set $w_3 = w_2 - (f(w_2)/2)v$. Then $a := f(w_3) = -g(w_3) > 0$. Hence, for any $t > 0$, we have $\|v + tw_3\| \geq f(v + tw_3) = 1 + ta$ while $\|v - tw_3\| \geq g(v - tw_3) = 1 + ta$. Thus we obtain

$$\frac{\rho_B(\tau)}{\tau} \geq \frac{\|v + \tau w_3\|}{2\tau} + \frac{\|v - \tau w_3\|}{2\tau} - \frac{1}{\tau} \geq a > 0$$

for all $\tau > 0$. Therefore B is not uniformly smooth.

If B is real, then the closed subspaces and the quotient spaces of B is also uniformly convex, and B is reflexive. The duality map j from the unit sphere of B into the unit sphere of B^* is uniformly continuous with a uniformly continuous inverse. The following proposition for the case that B is real is Proposition A.5. in [BL00].

Proposition 1.3.5. *Let B be a uniformly smooth Banach space.*

$$\| \operatorname{Re} j(v) - \operatorname{Re} j(u) \|_{B^*} \leq 2\rho_B \left(2 \left\| \frac{v}{\|v\|} - \frac{u}{\|u\|} \right\| \right) \Big/ \left\| \frac{v}{\|v\|} - \frac{u}{\|u\|} \right\|$$

for all $v, u \in B \backslash \{0\}$ with $v \neq u$. Here, for $f \in B^*$, we denote by $\operatorname{Re} f$ *the real-valued part of f. If B is real, then* Re *is trivial.*

Proof. For a complex number c, we denote by $\operatorname{Re} c$ the real part of c. Note that Re is linear. For $u \in B \backslash \{0\}$ and $v \in B$, we have

$$\operatorname{Re}(j(u)v) + \|u\| = \operatorname{Re}(j(u)(v+u)) \leq |j(u)(v+u)| \leq \|v+u\|.$$

Hence $\operatorname{Re}(j(u)v) \leq \|u+v\| - \|u\|$.

Fix $x, y \in B \backslash \{0\}$ with $x \neq y$. Since any $u \in B \backslash \{0\}$ satisfies $j(u) = j(u/\|u\|)$, we may assume that $\|x\| = \|y\| = 1$. Take an arbitrary $z \in B$ with $\|z\| = \|x-y\|$.

Then

$$
\begin{aligned}
(\operatorname{Re} j(y) - \operatorname{Re} j(x))z &= \operatorname{Re}(j(y)z) - \operatorname{Re}(j(x)z) \\
&\leq \|y+z\| - \|y\| - \operatorname{Re}(j(x)z) + \|x\| - \operatorname{Re}(j(x)y) \\
&= \|y+z\| - 1 + \operatorname{Re}(j(x)(x-y-z)) \\
&\leq \|y+z\| - 1 + \|x + (x-y-z)\| - \|x\| \\
&= \|x + (y-x+z)\| + \|x - (y-x+z)\| - 2 \\
&\leq 2\rho_B(\|y-x+z\|) \\
&\leq 2\rho_B(2\|y-x\|),
\end{aligned}
$$

because ρ_B is nondecreasing and $\|y-x+z\| \leq 2\|y-x\|$. Since z was arbitrary, the proposition follows. □

A real Banach space is uniformly smooth if and only if the dual Banach space is uniformly convex. Hence the class of uniformly convex and uniformly smooth real Banach spaces is closed under taking duals.

Example 1.3.6 ([Han56])**.** The modulus of convexity of L^p can be written as

$$
\delta_{L^p}(\epsilon) = \begin{cases} (p-1)\epsilon^2/8 + o(\epsilon^2) & \text{if } 1 < p \leq 2 \\ \epsilon^p/p2^p + o(\epsilon^p) & \text{if } 2 \leq p < \infty \end{cases}
$$

for $\epsilon > 0$, and the modulus of smoothness of L^p can be written as

$$
\rho_{L^p}(\tau) = \begin{cases} \tau^p/p + o(\tau^p) & \text{if } 1 < p \leq 2 \\ (p-1)\tau^2/2 + o(\tau^2) & \text{if } 2 \leq p < \infty \end{cases}
$$

for $\tau > 0$. In particular, for a Hilbert space H, $\delta_H(\epsilon) = 1 - (1 - \epsilon^2/4)^{1/2}$ for $\epsilon > 0$ and $\rho_H(\tau) = (1+\tau^2)^{1/2} - 1$ for $\tau > 0$.

Chapter 2

Isometric actions

In this chapter, we investigate an isometric action α of a finitely generated (or compactly generated) group on a global Busemann NPC space or a Banach space. To find a global fixed point of α, we will introduce a nonnegative continuous function $F_{\alpha,p}$ on the space which plays the most important role in this thesis, and investigate the behavior of $F_{\alpha,p}$ using its absolute gradient.

2.1 Global fixed points

A group G is said to be a *topological group* if G has a topology such that the map $G \times G \to G$, $(g, g') \mapsto gg'$ and the map on G, $g \mapsto g^{-1}$ are continuous. For example, topological vector spaces and Lie groups are topological groups. A topological group G is said to be *compactly generated* if there exists a compact subset $K \subset G$ such that any $g \in G$ is written as $g_1 g_2 \cdots g_k$ using some $g_1, g_2, \ldots, g_k \in K \cup K^{-1}$, where $K^{-1} = \{g^{-1} \in G : g \in K\}$. For instance, finite dimensional normed linear spaces, compact Lie groups and special linear groups $SL_n(\mathbb{R})$ are compactly generated groups. We call K a *compact generating subset* of G. If $K^{-1} = K$, then K is said to be symmetric. For a compactly generated group G, we can take a symmetric compact generating subset K. Indeed, for a compact generating subset K, $K \cup K^{-1}$ is a symmetric compact generating subset of G. Let μ be a measure on the topological σ-algebra of K, that is, the smallest σ-algebra of K which contains all open subsets of K. We suppose that μ is probability, that is, $\mu(K) = 1$, and the support is K, that is, every open neighborhood of each point in K has positive measure. We assume that every compactly generated group in this thesis has such a measure μ. An example of such a measure on a finite dimensional compactly generated Lie group G is constructed as follows: The group G has a left Haar measure. The restriction of the measure on a compact generating subset K is a measure on the topological

σ-algebra on K. Normalizing it, we obtain a probability measure supported on K.

A *discrete group* is a topological group with discrete topology. If a discrete group is compactly generated, then it is called a *finitely generated group*. For example, free groups, free Abelian groups and special linear groups $SL_n(\mathbb{Z})$ are finitely generated groups. A compact generating subset of a finitely generated group is a finite subset, hence we call it a *finite generating subset*. For a finite generating subset K of a finitely generated group, a positive function m on K satisfying $\sum_{\gamma \in K} m(\gamma) = 1$ can be regarded as a probability measure supported on K. We call it a *weight* on K. Note that every probability measure supported on a finite generating subset is identified with a certain weight on the subset. A weight m on a symmetric finite generating subset K is said to be *symmetric* if it satisfies $m(\gamma) = m(\gamma^{-1})$ for all $\gamma \in K$. Obviously, a finite generating subset always has a weight. We always denote by G a compactly generated group and by Γ a finitely generated group.

Let (M, d) be a metric space. An *isometry* on (M, d) is a map $f : M \to M$ such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$. The composition of a shortest geodesic and an isometry is a shortest geodesic. Hence we can say that an isometry maps a shortest geodesic to a shortest geodesic. The *isometry group* $\mathrm{Isom}(M, d)$ is the group consisting of all surjective isometries on (M, d). Let $\alpha : G \to \mathrm{Isom}(M, d)$ be a strongly continuous homomorphism. Here strongly continuous means that the map $G \to M$, $g \mapsto \alpha(g)(x)$ is continuous for each $x \in M$. We can regard α as an isometric action of G on M, hence we call α an *isometric action*. We write $\alpha(g)x$ as $\alpha(g)(x)$ for all $g \in G$ and $x \in M$. We say α has a *global fixed point* $x_0 \in M$ if $\alpha(g)x_0 = x_0$ for all $g \in G$.

Definition 2.1.1. We define $F_{\alpha,p} : M \to [0, \infty)$ by

$$\begin{aligned} F_{\alpha,p}(x) &:= \left(\int_{g \in K} d(x, \alpha(g)x)^p d\mu(g) \right)^{1/p} \text{ for } 1 \leq p < \infty, \\ F_{\alpha,\infty}(x) &:= \max_{g \in K} d(x, \alpha(g)x) \end{aligned}$$

for each $x \in M$. In particular, for a finitely generated group, we can write

$$F_{\alpha,p}(x) = \left(\sum_{\gamma \in K} d(x, \alpha(\gamma)x)^p m(\gamma) \right)^{1/p}$$

for each $1 \leq p < \infty$ and $x \in M$.

For any $x, y \in M$ and $g \in G$, we have

$$|d(x, \alpha(g)x) - d(y, \alpha(g)y)| \leq d(x, y) + d(\alpha(g)x, \alpha(g)y) = 2d(x, y).$$

Hence $d(x, \alpha(g)x)$ is continuous in x. Because of the definition of μ and the continuity of $d(x, \alpha(g)x)$ in g, $F_{\alpha,p}$ vanishes at $x_0 \in M$ if and only if x_0 is a global fixed point of α. Using Minkowski's inequality, for $1 \le p < \infty$, we have

$$\begin{aligned} F_{\alpha,p}(x) &= \left(\int_{g\in K} d(x, \alpha(g)x)^p d\mu(g) \right)^{1/p} \\ &\le \left(\int_{g\in K} (d(x, y) + d(y, \alpha(g)y) + d(\alpha(g)y, \alpha(g)x))^p d\mu(g) \right)^{1/p} \\ &\le \left(\int_{g\in K} d(y, \alpha(g)y)^p d\mu(g) \right)^{1/p} + 2 \left(\int_{g\in K} d(x, y)^p d\mu(g) \right)^{1/p} \\ &\le F_{\alpha,p}(y) + 2d(x, y) \end{aligned}$$

for all $x, y \in M$. Hence we obtain $|F_{\alpha,p}(x) - F_{\alpha,p}(y)| \le 2d(x, y)$, that is, $F_{\alpha,p}$ is continuous for $1 \le p < \infty$. We can easily show $|F_{\alpha,\infty}(x) - F_{\alpha,\infty}(y)| \le 2d(x, y)$ for $x, y \in N$, hence $F_{\alpha,\infty}$ is also continuous.

A function F on a geodesic space M is said to be *convex* if, for any shortest geodesic $c : [0, l] \to M$, $F(c(tl)) \le (1 - t)F(c(0)) + tF(c(l))$ for $t \in [0, 1]$. For an isometric action α on a global Busemann NPC space (N, d) and a shortest geodesic $c : [0, l] \to N$, by Theorem 1.1.3 and Minkowski's inequality, we have

$$\begin{aligned} & F_{\alpha,p}(c(tl)) \\ =\ & \left(\int_K d(c(tl), \alpha(g)c(tl))^p d\mu(g) \right)^{1/p} \\ \le\ & \left(\int_K ((1 - t)d(c(0), \alpha(g)c(0)) + td(c(l), \alpha(g)c(l)))^p d\mu(g) \right)^{1/p} \\ \le\ & \left(\int_K ((1 - t)d(c(0), \alpha(g)c(0)))^p d\mu(g) \right)^{1/p} \\ & \qquad\qquad + \left(\int_K (td(c(l), \alpha(g)c(l)))^p d\mu(g) \right)^{1/p} \\ =\ & (1 - t)F_{\alpha,p}(c(0)) + tF_{\alpha,p}(c(l)) \end{aligned}$$

for $1 \le p < \infty$. Therefore $F_{\alpha,p}$ is convex for $1 \le p < \infty$. The function $F_{\alpha,\infty}$ is also convex by an easy computation.

Lemma 2.1.2. *Let Γ be a finitely generated group, ω a non-principal ultrafilter on $\mathbb{N}$, and $1 \le p \le \infty$. Let $\{(M_n, d_n, o_n)\}$ be a sequence of metric spaces with base points and α_n an isometric action of Γ on (M_n, d_n) for each n. Suppose that $F_{\alpha_n,p}(o_n)$ is bounded independently of n. Then we can define an isometric action α_ω of Γ on (M_ω, d_ω) by $\alpha_\omega(\gamma)x_\omega := \omega\text{-}\lim_n(\alpha_n(\gamma)x_n)$ for all $\gamma \in \Gamma$ and*

$x_\omega \in M_\omega$, where $(x_n) \in M_\infty$ is a representative of x_ω. Moreover, $F_{\alpha_\omega,p}(x_\omega) = \omega\text{-}\lim_n F_{\alpha_n,p}(x_n)$ for all $x_\omega \in M_\omega$ and their representatives $(x_n) \in M_\infty$.

Proof. Since $\{F_{\alpha_n,p}(o_n)\}$ is bounded independently of n, $\{d(o_n, \alpha_n(\gamma)o_n)\}$ is also bounded independently of n for all $\gamma \in K$. Since any $(x_n) \in M_\infty$ satisfies

$$\begin{aligned} d_n(o_n, \alpha_n(\gamma)x_n) &\leq d_n(o_n, \alpha_n(\gamma)o_n) + d_n(\alpha_n(\gamma)o_n, \alpha_n(\gamma)x_n) \\ &= d_n(o_n, \alpha_n(\gamma)o_n) + d_n(o_n, x_n) \end{aligned}$$

for all $\gamma \in \Gamma$, n, we have $(\alpha_n(\gamma)x_n) \in M_\infty$. For $x_\omega, y_\omega \in M_\omega$, take arbitrary representatives $(x_n), (y_n) \in M_\infty$ respectively. Then, for each $\gamma \in \Gamma$, we have

$$\begin{aligned} d_\omega(\alpha_\omega(\gamma)x_\omega, \alpha_\omega(\gamma)y_\omega) &= \omega\text{-}\lim_n d_n(\alpha_n(\gamma)x_n, \alpha_n(\gamma)y_n) \\ &= \omega\text{-}\lim_n d_n(x_n, y_n) \\ &= d_\omega(x_\omega, y_\omega). \end{aligned}$$

Hence $\alpha_\omega(\gamma)$ is well-defined and is an isometry on (M_ω, d_ω) for each $\gamma \in \Gamma$. For $\gamma_1, \gamma_2 \in \Gamma$, $x_\omega \in M_\omega$, and a representative $(x_n) \in M_\infty$ of x_ω,

$$d_\omega(\alpha_\omega(\gamma_1\gamma_2)x_\omega, \alpha_\omega(\gamma_1)\alpha_\omega(\gamma_2)x_\omega) = \omega\text{-}\lim_n d_n(\alpha_n(\gamma_1\gamma_2)x_n, \alpha_n(\gamma_1)\alpha_n(\gamma_2)x_n) = 0,$$

hence we get $\alpha_\omega(\gamma_1\gamma_2) = \alpha_\omega(\gamma_1)\alpha_\omega(\gamma_2)$. Thus $\alpha_\omega(\gamma)\alpha_\omega(\gamma^{-1})$ is the identity mapping. Hence each $\alpha_\omega(\gamma)$ is surjective. Moreover, α_ω is a homomorphism from Γ into $\mathrm{Isom}(M_\omega, d_\omega)$. Therefore α_ω is an isometric action. Since, for each $(x_n) \in M_\infty$, $\{d(x_n, \alpha(\gamma)x_n)\}$ is bounded independently of n, $\{F_{\alpha_n,p}(x_n)\}$ is also bounded independently of n. In the case that $1 \leq p < \infty$, using Lemma 1.2.2 and Lemma 1.2.4, and in the case that $p = \infty$ using Lemma 1.2.5, we have $F_{\alpha_\omega,p}(x_\omega) = \omega\text{-}\lim_n F_{\alpha_n,p}(x_n)$ for all $x_\omega \in M_\omega$ and their representatives $(x_n) \in M_\infty$. □

2.2 Absolute gradient

Throughout this section, let (M, d) be a complete geodesic space, and F a convex nonnegative lower semicontinuous function on M. In this section, we define the absolute gradient of F on (M, d), and show some of its properties.

Definition 2.2.1. We define the *absolute gradient* $|\nabla_- F|$ of F at $x \in M$ by

$$|\nabla_- F|(x) := \max\left\{ \limsup_{y \to x,\, y \in M} \frac{F(x) - F(y)}{d(x, y)},\ 0 \right\}.$$

The following Proposition 2.2.2, Corollary 2.2.3 and Proposition 2.2.4 was proved by Mayer [May98], when (M, d) is a Hadamard space. His proofs are valid for complete geodesic spaces.

Proposition 2.2.2 ([May98], Proposition 2.34)**.**

$$|\nabla_{-}F|(x) = \max\left\{ \sup_{y\neq x,\, y\in M} \frac{F(x)-F(y)}{d(x,y)},\, 0 \right\}$$

at all $x \in M$.

Proof. Let x be an arbitrary point in M. It suffices to show that

$$\sup_{y\neq x,\, y\in M} \frac{F(x)-F(y)}{d(x,y)} \leq \limsup_{y\to x,\, y\in M} \frac{F(x)-F(y)}{d(x,y)}.$$

Let $y \in M\backslash\{x\}$ and $c : [0, l] \to M$ be a shortest geodesic joining x to y. From the convexity of F, we get $t(F(x) - F(y)) \leq F(x) - F(c(tl))$ for all $t \in [0, 1]$. Since $td(x, y) = d(x, c(tl))$, we have

$$\frac{F(x)-F(y)}{d(x,y)} \leq \limsup_{t\to 0} \frac{F(x)-F(c(tl))}{d(x,c(tl))}.$$

Because we can take y arbitrarily, we obtain

$$\begin{aligned} \sup_{y\neq x,\, y\in M} \frac{F(x)-F(y)}{d(x,y)} &\leq \sup_{y\neq x,\, y\in M} \left(\limsup_{t\to 0} \frac{F(x)-F(c(tl))}{d(x,c(tl))} \right) \\ &\leq \limsup_{y\to x,\, y\in M} \frac{F(x)-F(y)}{d(x,y)}. \end{aligned}$$

□

We suppose that $|\nabla_{-}F|(x) < \infty$ for all $x \in M$. Since $F_{\alpha,p}$ in Section 2.1 satisfies $|F_{\alpha,p}(x) - F_{\alpha,p}(y)| \leq 2d(x, y)$ for all $x, y \in M$, we have

$$|\nabla_{-}F_{\alpha,p}|(x) \leq \limsup_{y\to x,\, y\in \mathcal{M}} \frac{|F_{\alpha,p}(x)-F_{\alpha,p}(y)|}{d(x,y)} \leq 2$$

for all $x \in M$. In particular, $|\nabla_{-}F_{\alpha,p}|(x) < \infty$ for all $x \in M$.

Corollary 2.2.3 ([May98], Corollary 2.35)**.** *A point $x_0 \in M$ minimizes F if and only if $|\nabla_{-}F|$ vanishes at x_0.*

Proof. If $|\nabla_{-}F|(x_0) = 0$, by Proposition 2.2.2, we have $F(x_0) \leq F(y)$ for all $y \in M$. Hence x_0 minimizes F. The converse is obvious. □

Proposition 2.2.4 ([May98], Proposition 2.25)**.** *The absolute gradient* $|\nabla_- F|$ *is lower semicontinuous.*

Proof. Since $|\nabla_- F|$ is nonnegative, it is lower semicontinuous at all $x \in M$ such that $|\nabla_- F|(x) = 0$. Consider $x \in M$ such that $|\nabla_- F|(x) > 0$. For $\epsilon > 0$ and $\delta > 0$ with $|\nabla_- F|(x) > \delta$, we can take $y \in M$ such that $F(x) - F(y) > \delta d(x, y)$ and $d(x, y) < \epsilon$. From the lower semicontinuity of F at x, we can take $z \in M$ such that $d(x, z) < \epsilon d(x, y)$ and $F(x) - F(z) \leq \delta\epsilon d(x, y)$. Let $c : [0, l] \to M$ be a shortest geodesic joining z to y. By the convexity of F, we have $F(z) - F(c(tl)) \geq t(F(z) - F(y))$. Since $d(z, c(tl)) = td(z, y)$, we get

$$\begin{aligned}
|\nabla_- F|(z) &\geq \limsup_{t\to 0} \frac{F(z) - F(c(tl))}{d(z, c(tl))} \\
&\geq \frac{F(z) - F(y)}{d(z, y)} \\
&= \frac{F(z) - F(x)}{d(z, y)} + \frac{F(x) - F(y)}{d(z, y)} \\
&> \frac{-\delta\epsilon d(x, y)}{d(z, y)} + \frac{\delta d(x, y)}{d(z, y)} = \frac{\delta(1 - \epsilon)d(x, y)}{d(z, y)}.
\end{aligned}$$

Since $d(z, y) \leq d(z, x) + d(x, y) < (1 + \epsilon)d(x, y)$, we obtain

$$\frac{\delta(1 - \epsilon)d(x, y)}{d(z, y)} > \frac{\delta(1 - \epsilon)}{1 + \epsilon} > \delta(1 - 2\epsilon).$$

Therefore we have $|\nabla_- F|(z) > \delta(1 - 2\epsilon)$ for all $z \in M$ sufficiently close to x. For the fixed δ, if z approaches x, then we can take ϵ smaller. Hence we obtain $\liminf_{z\to x} |\nabla_- F|(z) \geq \delta$. Since δ with $|\nabla_- F|(x) > \delta$ is arbitrary, we have $\liminf_{z\to x} |\nabla_- F|(z) \geq |\nabla_- F|(x)$. Therefore $|\nabla_- F|$ is lower semicontinuous. □

Lemma 2.2.5 ([Tan])**.** $\inf_{x\in M} |\nabla_- F|(x) = 0$.

Proof. Suppose that $C := \inf_{x\in M} |\nabla_- F|(x) > 0$. Then

$$\sup_{y\neq x,\, y\in M} \frac{F(x) - F(y)}{d(x, y)} \geq C$$

for all $x \in M$ by Proposition 2.2.2. In particular, F is positive. Let $x_0 \in M$ and $0 < \epsilon < 1$, and set

$$A_0 := \left\{ y \in M\backslash\{x_0\} : (1 - \epsilon)C \leq \frac{F(x_0) - F(y)}{d(x_0, y)} \right\}.$$

By the definition of C, we have $A_0 \neq \emptyset$.

We may assume that $\inf_{y\in A_0} F(y) > 0$. Indeed, set $F'(x) := F(x) + 1$ for all $x \in M$. Then F' is nonnegative and lower semicontinuous. Since, for any shortest geodesic $c : [0, l] \to N$,

$$F'(c(tl)) \leq (1-t)F(c(0)) + tF(c(l)) + 1 = (1-t)F'(c(0)) + tF'(c(l))$$

for all $t \in [0, 1]$, F' is convex. Furthermore, $|\nabla_- F|(x) = |\nabla_- F'|(x)$ for all $x \in N$ by the definition of absolute gradient. Hence, if $\inf_{y\in A_0} F(y) = 0$, then, replacing F with F', we can assume $\inf_{y\in A_0} F(y) > 0$.

Take $x_1 \in A_0$ such that $F(x_1) \leq (1+\epsilon)\inf_{y\in A_0} F(y)$, and set

$$A_1 := \left\{ y \in M\backslash\{x_1\} : (1-\epsilon)C \leq \frac{F(x_1) - F(y)}{d(x_1, y)} \right\}.$$

By the definition of C, we have $A_1 \neq \emptyset$. As $F(x_1) < F(x_0)$, $x_0 \notin A_1$. Since $x_1 \in A_0$, for any $y \in A_1$ we obtain

$$\begin{aligned}
\frac{F(x_0) - F(y)}{d(x_0, y)} &\geq \frac{(F(x_0) - F(x_1)) + (F(x_1) - F(y))}{d(x_0, x_1) + d(x_1, y)} \\
&\geq \frac{(1-\epsilon)Cd(x_0, x_1) + (1-\epsilon)Cd(x_1, y)}{d(x_0, x_1) + d(x_1, y)} \\
&= (1-\epsilon)C.
\end{aligned}$$

Hence $y \in A_0$, that is, $A_1 \subset A_0$. Thus $\inf_{y\in A_1} F(y) > 0$. Inductively, for each $i \in \mathbb{N}$, take $x_i \in A_{i-1}$ such that $F(x_i) \leq (1+\epsilon^i)\inf_{y\in A_{i-1}} F(y)$, and set

$$A_i := \left\{ y \in M\backslash\{x_i\} : (1-\epsilon)C \leq \frac{F(x_i) - F(y)}{d(x_i, y)} \right\}.$$

We can also show that $A_i \neq \emptyset$, $A_i \subset A_{i-1}$, and $\inf_{y\in A_i} F(y) > 0$ for each i. Thus, for $y \in A_i$ we have

$$\begin{aligned}
d(x_i, y) &\leq \frac{F(x_i) - F(y)}{(1-\epsilon)C} \\
&\leq \frac{(1+\epsilon^i)\inf_{z\in A_{i-1}} F(z) - \inf_{z'\in A_i} F(z')}{(1-\epsilon)C} \\
&\leq \frac{(1+\epsilon^i)\inf_{z\in A_{i-1}} F(z) - \inf_{z\in A_{i-1}} F(z)}{(1-\epsilon)C} \\
&= \frac{\epsilon^i \inf_{z\in A_{i-1}} F(z)}{(1-\epsilon)C}.
\end{aligned}$$

Since $x_i \in A_{i-1}$ and $F(x_i) \leq F(x_{i-1})$ for each i, we have

$$d(x_i, y) \leq \frac{\epsilon^i F(x_i)}{(1-\epsilon)C} \leq \frac{\epsilon^i F(x_0)}{(1-\epsilon)C}$$

for all $y \in A_i$. Thus for any $\epsilon' > 0$ there exists j such that

$$\operatorname{diam} A_i \leq 2\frac{\epsilon^i F(x_0)}{(1-\epsilon)C} < \epsilon'$$

for all $i \geq j$, where $\operatorname{diam} A_i$ is the diameter of A_i. Furthermore, $\{x_i\}_{i\in\mathbb{N}} \subset M$ satisfies $d(x_j, x_k) \leq \operatorname{diam} A_i$ for all $j, k \geq i$. Since M is complete, $\{x_i\}_{i\in\mathbb{N}}$ converges to some $x_\infty \in M$. Thus $\bigcap_{i=0}^{\infty}(A_i \cup \{x_i\})$ is either the one-point set $\{x_\infty\}$ or $\emptyset$.

By the way, $1/d(x_i, y)$ is continuous on the open set $M\backslash\{x_i\}$, and $-F$ is upper semicontinuous on M. Thus

$$F_i'(y) := \frac{F(x_i) - F(y)}{d(x_i, y)}$$

is upper semicontinuous on $M\backslash\{x_i\}$. Hence $\{y \in M\backslash\{x_i\} : F_i'(y) < r\}$ is open for any $r > 0$. Since

$$\begin{aligned}\{y \in M\backslash\{x_i\} : F_i'(y) < (1-\epsilon)C\} &= M\backslash(\{y \in M \ : (1-\epsilon)C \leq F_i'(y)\} \cup \{x_i\}) \\ &= M\backslash(A_i \cup \{x_i\}),\end{aligned}$$

$A_i \cup \{x_i\}$ is closed for all i. This implies that $\bigcap_{i=0}^{\infty}(A_i \cup \{x_i\}) = \{x_\infty\}$. However, by the assumption that $C > 0$, there exists $y_0 \in M\backslash\{x_\infty\}$ such that

$$(1-\epsilon)C \leq \frac{F(x_\infty) - F(y_0)}{d(x_\infty, y_0)}.$$

Since $x_\infty \in A_{i+1} \cup \{x_{i+1}\}$, we get $x_\infty \in A_i$ for all i. Hence $F(y_0) < F(x_\infty) < F(x_i)$ for all i. Thus $x_i \neq y_0$ and

$$\begin{aligned}\frac{F(x_i) - F(y_0)}{d(x_i, y_0)} &\geq \frac{(F(x_i) - F(x_\infty)) + (F(x_\infty) - F(y_0))}{d(x_i, x_\infty) + d(x_\infty, y_0)} \\ &\geq \frac{(1-\epsilon)Cd(x_i, x_\infty) + (1-\epsilon)Cd(x_\infty, y_0)}{d(x_i, x_\infty) + d(x_\infty, y_0)} \\ &= (1-\epsilon)C\end{aligned}$$

for all i. This implies that $y_0 \in \bigcap_{i=1}^{\infty}(A_i \cup \{x_i\}) = \{x_\infty\}$, that is, $x_\infty = y_0$. This contradicts the assumption that $y_0 \in M\backslash\{x_\infty\}$.

□

Corollary 2.2.6. *There exists $\{y_n\} \subset M$ such that $|\nabla_- F|(y_n) \to 0$ as $n \to \infty$. If $\{y_n\}$ converges to $y_\infty \in M$, then y_∞ minimizes F.*

Proof. By Lemma 2.2.5, the existence of such a sequence $\{y_n\} \subset M$ is obvious. If $\{y_n\}$ converges to $y_\infty \in M$, then, by Proposition 2.2.4, we have $|\nabla_- F|(y_\infty) \leq \lim_{n\to\infty} |\nabla_- F|(y_n) = 0$. It follows from Corollary 2.2.3 that y_∞ minimizes F. □

2.3 Affine isometric actions

In this section, we give the definitions of linear isometric representations and affine isometric actions of a finitely generated group, and the first cohomology with respect to a linear isometric representation. Furthermore, we will give explicit expression of $|\nabla_- F_{\alpha,p}|$ for an affine isometric action α on a strictly convex and smooth real Banach space and a uniformly convex and uniformly smooth Banach space. In particular, in Corollary 2.3.4, we will see that the expression of $|\nabla_- F_{\alpha,p}|$ matches with the affine isometric action on L^p with $1 < p < \infty$.

Let Γ be a finitely generated group, K a finite generating subset of Γ, m a weight on K, and $(B, \| \ \|)$ a Banach space. The *orthogonal group* $O(B)$ of B is the group of all surjective linear isometries on B. An isometry T on B is said to be *affine* if it is written as $Tv = Lv + u$ by some linear isometry L on B and $u \in B$. If we fix the origin of B, then this decomposition is unique. The *affine isometry group* $\mathrm{Aff}(B)$ of B is the group of all surjective affine isometries on B, that is, $\mathrm{Aff}(B) = O(B) \ltimes B$. A classical theorem of Mazur-Ulam says that every surjective isometry on a real Banach space is affine (see [BL00]). Hence, if B is real, $\mathrm{Aff}(B)$ coincides with $\mathrm{Isom}(B, \| \ \|)$.

A *linear isometric representation* of Γ on B is a homomorphism from Γ into $O(B)$. For example, the left regular representation $\lambda_{\Gamma,p}$ of Γ on $\ell^p(\Gamma)$ with respect to the uniform measure on Γ is defined by $\lambda_{\Gamma,p}(\gamma)f(\gamma') = f(\gamma^{-1}\gamma')$ for each $f \in \ell^p(\Gamma)$ and $\gamma, \gamma' \in \Gamma$, it is a linear isometric representation of Γ on $\ell^p(\Gamma)$. An *affine isometric action* of Γ on B is a homomorphism from Γ into $\mathrm{Aff}(B)$. For a linear isometric representation π, $c : \Gamma \to B$ is called a *π-cocycle* if it satisfies $c(\gamma\gamma') = \pi(\gamma)c(\gamma') + c(\gamma)$ for all $\gamma, \gamma' \in \Gamma$. A cocycle is completely determined by its values on K. An affine isometric action α has the form $\alpha(\gamma)v = \pi(\gamma)v + c(\gamma)$ for each $\gamma \in \Gamma$ and $v \in B$, where π is a linear isometric representation and c is a π-cocycle. We call π the *linear part* of α and c the *cocycle part* of α, and we write $\alpha = \pi + c$.

Let π be a linear isometric representation of Γ on B. We denote by $Z^1(\pi)$ the linear space consisting of all π-cocycles. We define a linear map $d : B \to Z^1(\pi)$ by $dv(\gamma) := v - \pi(\gamma)v$ for each $v \in B$ and $\gamma \in \Gamma$. Here, for $v \in B$,

$$dv(\gamma\gamma') = v - \pi(\gamma\gamma')v = (v - \pi(\gamma)v) + (\pi(\gamma)v - \pi(\gamma)\pi(\gamma')v) = dv(\gamma) + \pi(\gamma)dv(\gamma')$$

for all $\gamma, \gamma' \in \Gamma$, hence d is well-defined. We set $B^1(\pi) := d(B)$, and we call an element in $B^1(\pi)$ a *π-coboundary*. It is a linear subspace of $Z^1(\pi)$. If π has no non-trivial invariant vector, then d is an isomorphism from B onto $B^1(\pi)$.

The space $Z^1(\pi)$ describes the set of all affine isometric actions of Γ on B with the linear part π. Each π-coboundary corresponds to such an affine isometric action having a global fixed point. Indeed, if an affine isometric action $\alpha = \pi + c$ fixes $v \in B$, then we have $c(\gamma) = \alpha(\gamma)v - \pi(\gamma)v = v - \pi(\gamma)v = dv(\gamma)$ for all $\gamma \in \Gamma$. Hence c is dv. On the contrary, if c is dv, then $\alpha(\gamma)v = \pi(\gamma)v + dv(\gamma) = v$ for all $\gamma \in \Gamma$. Hence v is a global fixed point of α. *The first cohomology of* Γ *with respect to* π is $H^1(\Gamma, \pi) := Z^1(\pi)/B^1(\pi)$. Note that $H^1(\Gamma, \pi)$ vanishes if and only if every affine isometric action of Γ on B with the linear part π has a global fixed point.

We endow $Z^1(\pi)$ with the norm

$$\|c\|_p := \left(\sum_{\gamma \in K} \|c(\gamma)\|^p m(\gamma) \right)^{1/p}$$

for $1 \leq p < \infty$, or the norm

$$\|c\|_\infty := \max_{\gamma \in K} \|c(\gamma)\|.$$

Then $Z^1(\pi)$ becomes a Banach space with respect to each of these norms. Note that, in general, $B^1(\pi)$ is not closed in $Z^1(\pi)$. Since

$$\|dv\|_p = \left(\sum_{\gamma \in K} \|v - \pi(\gamma)v\|^p m(\gamma) \right)^{1/p} \leq \left(\sum_{\gamma \in K} (2\|v\|)^p m(\gamma) \right)^{1/p} = 2\|v\|$$

and $\|dv\|_\infty = \max_{\gamma \in K} \|v - \pi(\gamma)v\| \leq 2\|v\|$ for all $v \in B$, d is bounded with respect to each of these norms. For an affine isometric action $\alpha = \pi + c$ and $1 \leq p \leq \infty$, $F_{\alpha,p}(v)$ coincides with $\|dv - c\|_p$ at each $v \in B$.

Lemma 2.3.1. *Let F be a convex function on B. Then we have*

$$\limsup_{u \to v,\, u \in B} \frac{F(v) - F(u)}{\|v - u\|} = \sup_{u \in B; \|u\|=1} \lim_{\epsilon \to 0} \frac{F(v) - F(v + \epsilon u)}{\epsilon}.$$

Proof. Since F is convex, for $t \geq s > 0$, we have

$$F(v + su) \leq \left(1 - \frac{s}{t}\right) F(v) + \frac{s}{t} F(v + tu).$$

Hence we have

$$\frac{F(v) - F(v + tu)}{t} \leq \frac{F(v) - F(v + su)}{s}.$$

This implies that

$$\lim_{\epsilon \to 0} \frac{F(v) - F(v + \epsilon u)}{\epsilon} = \sup_{s > 0} \frac{F(v) - F(v + su)}{s}.$$

This completes the proof. □

Proposition 2.3.2. *Suppose that K is symmetric, m is symmetric, and B is strictly convex, smooth and real. Let $\alpha = \pi + c$ be an affine isometric action of Γ on B, and $1 < p < \infty$. For $v \in B$ such that $F_{\alpha,p}(v) > 0$, we have*

$$\begin{aligned}
&|\nabla_- F_{\alpha,p}|(v) \\
=\ & \frac{1}{F_{\alpha,p}(v)^{p-1}} \sup_{v \in B; \|u\|=1} \sum_{\gamma \in K} \|v - \alpha(\gamma)v\|^{p-1} j(v - \alpha(\gamma)v)(u - \pi(\gamma)u) m(\gamma) \\
=\ & \frac{2}{F_{\alpha,p}(v)^{p-1}} \left\| \sum_{\gamma \in K} \|v - \alpha(\gamma)v\|^{p-1} m(\gamma) j(v - \alpha(\gamma)v) \right\|_{B^*}.
\end{aligned}$$

In particular, if α is linear, then $|\nabla_- F_{\alpha,p}|(v) \geq F_{\alpha,p}(v)/\|v\|$ for all $v \in B \backslash \{0\}$.

Proof. Let $v \in B$ such that $F_{\alpha,p}(v) > 0$. Since $F_{\alpha,p}$ is convex, by Lemma 2.3.1, we have

$$\limsup_{u \to v,\, u \in B} \frac{F_{\alpha,p}(v) - F_{\alpha,p}(u)}{\|v - u\|} = \sup_{u \in B; \|u\|=1} \lim_{\epsilon \to 0} \frac{F_{\alpha,p}(v) - F_{\alpha,p}(v + \epsilon u)}{\epsilon}.$$

To calculate the right hand side, we use an inequality in [HLP52, (2.15.1)]:

$$pb^{p-1}(a - b) \leq a^p - b^p \leq pa^{p-1}(a - b)$$

for $a, b > 0$. Set $T(\gamma)u := u - \alpha(\gamma)u$ and $T_0(\gamma)u := u - \pi(\gamma)u$ for each $\gamma \in K$ and $u \in B$. Then, for a small $\epsilon > 0$, we have

$$\begin{aligned}
& \frac{F_{\alpha,p}(v) - F_{\alpha,p}(v + \epsilon u)}{\epsilon} \\
\leq\ & \frac{F_{\alpha,p}(v)^p - F_{\alpha,p}(v + \epsilon u)^p}{pF_{\alpha,p}(v + \epsilon u)^{p-1}\epsilon} \\
=\ & \sum_{\gamma \in K} \frac{\|T(\gamma)v\|^p - \|T(\gamma)(v + \epsilon u)\|^p}{pF_{\alpha,p}(v + \epsilon u)^{p-1}\epsilon} m(\gamma) \\
\leq\ & \sum_{\gamma \in K} \left(\frac{\|T(\gamma)v\|^{p-1}}{F_{\alpha,p}(v + \epsilon u)^{p-1}} \frac{\|T(\gamma)v\| - \|T(\gamma)(v + \epsilon u)\|}{\epsilon} \right) m(\gamma).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \frac{F_{\alpha,p}(v) - F_{\alpha,p}(v + \epsilon u)}{\epsilon} \\
\geq\ & \sum_{\gamma \in K} \left(\frac{\|T(\gamma)(v + \epsilon u)\|^{p-1}}{F_{\alpha,p}(v)^{p-1}} \frac{\|T(\gamma)v\| - \|T(\gamma)(v + \epsilon u)\|}{\epsilon} \right) m(\gamma).
\end{aligned}$$

Since B is smooth, for $\gamma \in K$ such that $T(\gamma)v \neq 0$,

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{\|T(\gamma)v\| - \|T(\gamma)(v + \epsilon u)\|}{\epsilon} &= \lim_{\epsilon \to 0} \frac{\|T(\gamma)v\| - \|T(\gamma)v + \epsilon T_0(\gamma)u\|}{\epsilon} \\
&= -j(T(\gamma)v)(T_0(\gamma)u),
\end{aligned}$$

and, for $\gamma \in K$ such that $T(\gamma)v = 0$,

$$\lim_{\epsilon \to 0} \frac{\|T(\gamma)v\| - \|T(\gamma)(v + \epsilon u)\|}{\epsilon} = \lim_{\epsilon \to 0} \frac{-\|\epsilon T_0(\gamma)u\|}{\epsilon} = -\|T_0(\gamma)u\|.$$

Therefore, by the assumption that $1 < p < \infty$, we have

$$\begin{aligned} & \lim_{\epsilon \to 0} \sum_{\gamma \in K} \left(\frac{\|T(\gamma)v\|^{p-1}}{F_{\alpha,p}(v + \epsilon u)^{p-1}} \frac{\|T(\gamma)v\| - \|T(\gamma)(v + \epsilon u)\|}{\epsilon} \right) m(\gamma) \\ = & \lim_{\epsilon \to 0} \sum_{\gamma \in K} \left(\frac{\|T(\gamma)(v + \epsilon u)\|^{p-1}}{F_{\alpha,p}(v)^{p-1}} \frac{\|T(\gamma)v\| - \|T(\gamma)(v + \epsilon u)\|}{\epsilon} \right) m(\gamma) \\ = & -\sum_{\gamma \in K} \frac{\|T(\gamma)v\|^{p-1}}{F_{\alpha,p}(v)^{p-1}} (j(T(\gamma)v)T_0(\gamma)u)m(\gamma) \\ = & \frac{-1}{F_{\alpha,p}(v)^{p-1}} \sum_{\gamma \in K} \|T(\gamma)v\|^{p-1} (j(T(\gamma)v)u - j(T(\gamma)v)\pi(\gamma)u)m(\gamma). \end{aligned}$$

If the last line of this equality is nonnegative, then the first equality in the proposition is proved. To prove this, we continue the computation of the last line of this equality. For arbitrary $\gamma \in K$, since $\pi(\gamma)$ is a surjective linear isometry,

$$\|j(T(\gamma)v)\pi(\gamma)\|_{B^*} = \|j(T(\gamma)v)\|_{B^*} = 1$$

and

$$(j(T(\gamma)v)\pi(\gamma))\, \pi(\gamma^{-1})T(\gamma)v = \|T(\gamma)v\| = \|\pi(\gamma^{-1})T(\gamma)v\|.$$

Due to the smoothness of B, these equalities imply that $j(T(\gamma)v)\pi(\gamma)$ coincides with $j(\pi(\gamma^{-1})T(\gamma)v)$. Since $c(e) = c(ee) = \pi(e)c(e) + c(e) = 2c(e)$ for the identity element e of Γ, $c(e)$ is trivial. Hence we have $\pi(\gamma^{-1})c(\gamma) + c(\gamma^{-1}) = c(\gamma^{-1}\gamma) = 0$ for all $\gamma, \gamma' \in \Gamma$. Since

$$\begin{aligned} \pi(\gamma^{-1})T(\gamma)v &= \pi(\gamma^{-1})v - \pi(\gamma^{-1})\alpha(\gamma)v \\ &= \pi(\gamma^{-1})v - \pi(\gamma^{-1})\pi(\gamma)v - \pi(\gamma^{-1})c(\gamma) \\ &= \pi(\gamma^{-1})v - v + c(\gamma^{-1}) \\ &= -T(\gamma^{-1})v, \end{aligned}$$

we get $j(T(\gamma)v)\pi(\gamma) = j(-T(\gamma^{-1})v) = -j(T(\gamma^{-1})v)$. Because

$$\|T(\gamma^{-1})v\| = \|\pi(\gamma^{-1})T(\gamma)v\| = \|T(\gamma)v\|$$

and m is symmetric, we have

$$\begin{aligned}
&\sum_{\gamma\in K}\|T(\gamma)v\|^{p-1}(j(T(\gamma)v)u-j(T(\gamma)v)\pi(\gamma)u)m(\gamma)\\
=\ &\sum_{\gamma\in K}\|T(\gamma)v\|^{p-1}j((T(\gamma)v)u+j(T(\gamma^{-1})v)u)m(\gamma)\\
=\ &\sum_{\gamma\in K}\|T(\gamma)v\|^{p-1}(j(T(\gamma)v)u)m(\gamma)+\sum_{\gamma\in K}\|T(\gamma^{-1})v\|^{p-1}(j(T(\gamma^{-1})v)u)m(\gamma^{-1})\\
=\ &2\sum_{\gamma\in K}\|T(\gamma)v\|^{p-1}(j(T(\gamma)v)u)m(\gamma).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\limsup_{u\to v,\,u\in B}\frac{F_{\alpha,p}(v)-F_{\alpha,p}(u)}{\|v-u\|}\\
=\ &\sup_{u\in B;\|u\|=1}\frac{2}{F_{\alpha,p}(v)^{p-1}}\left(-\sum_{\gamma\in K}\|T(\gamma)v\|^{p-1}m(\gamma)j(T(\gamma)v)\right)u\\
=\ &\frac{2}{F_{\alpha,p}(v)^{p-1}}\left\|\sum_{\gamma\in K}\|T(\gamma)v\|^{p-1}m(\gamma)j(T(\gamma)v)\right\|_{B^*}.
\end{aligned}$$

Since the last line of the above equality is nonnegative, we have

$$|\nabla_{-}F_{\alpha,p}|(v)=\limsup_{u\to v,\,u\in B}\frac{F_{\alpha,p}(v)-F_{\alpha,p}(u)}{\|v-u\|}.$$

If α is linear, then $\alpha=\pi$. Substituting $v/\|v\|$ for u in

$$\frac{1}{F_{\alpha,p}(v)^{p-1}}\sum_{\gamma\in K}\|v-\alpha(\gamma)v\|^{p-1}j(v-\alpha(\gamma)v)(u-\pi(\gamma)u)m(\gamma),$$

we obtain $|\nabla_{-}F_{\alpha,p}|(v)\geq F_{\alpha,p}(v)/\|v\|$ for any $v\in B\backslash\{0\}$. □

Proposition 2.3.3. *Suppose that K is symmetric, m is symmetric, and B is uniformly convex and uniformly smooth. Let $\alpha=\pi+c$ be an affine isometric action of Γ on B, and $1<p<\infty$. Then, for $v\in B$ with $F_{\alpha,p}(v)>0$, we have*

$$\begin{aligned}
&|\nabla_{-}F_{\alpha,p}|(v)\\
=\ &\frac{1}{F_{\alpha,p}(v)^{p-1}}\sup_{v\in B;\|u\|=1}\sum_{\gamma\in K}\|v-\alpha(\gamma)v\|^{p-1}\operatorname{Re}j(v-\alpha(\gamma)v)(u-\pi(\gamma)u)m(\gamma)\\
=\ &\frac{2}{F_{\alpha,p}(v)^{p-1}}\left\|\sum_{\gamma\in K}\|v-\alpha(\gamma)v\|^{p-1}m(\gamma)\operatorname{Re}j(v-\alpha(\gamma)v)\right\|_{B^*}.
\end{aligned}$$

Here $\operatorname{Re}j(u)$ is the real-valued part of $j(u)$ for $u\in B$.

Proof. Set $T(\gamma)u := u - \alpha(\gamma)u$ and $T_0(\gamma)u := u - \pi(\gamma)u$ for each $u \in B$ and $\gamma \in K$. These are continuous on B. Let $v \in B$ such that $F_{\alpha,p}(v) > 0$. As Proposition 2.3.2, for a small $\epsilon > 0$ we have

$$\begin{aligned} & \frac{F_{\alpha,p}(v) - F_{\alpha,p}(v+\epsilon u)}{\epsilon} \\ \leq & \sum_{\gamma \in K} \left(\frac{\|T(\gamma)v\|^{p-1}}{F_{\alpha,p}(v+\epsilon u)^{p-1}} \frac{\|T(\gamma)v\| - \|T(\gamma)(v+\epsilon u)\|}{\epsilon} \right) m(\gamma), \end{aligned}$$

and

$$\begin{aligned} & \frac{F_{\alpha,p}(v) - F_{\alpha,p}(v+\epsilon u)}{\epsilon} \\ \geq & \sum_{\gamma \in K} \left(\frac{\|T(\gamma)(v+\epsilon u)\|^{p-1}}{F_{\alpha,p}(v)^{p-1}} \frac{\|T(\gamma)v\| - \|T(\gamma)(v+\epsilon u)\|}{\epsilon} \right) m(\gamma). \end{aligned}$$

For $\gamma \in K$ such that $T(\gamma)v \neq 0$, we get

$$\begin{aligned} \|T(\gamma)v\| - \|T(\gamma)(v+\epsilon u)\| & \leq \operatorname{Re} j(T(\gamma)v)(T(\gamma)v - T(\gamma)(v+\epsilon u)) \\ & = -\epsilon \operatorname{Re} j(T(\gamma)v)(T_0(\gamma)u), \end{aligned}$$

and

$$\begin{aligned} \|T(\gamma)v\| - \|T(\gamma)(v+\epsilon u)\| & \geq \operatorname{Re} j(T(\gamma)(v+\epsilon u))(T(\gamma)v - T(\gamma)(v+\epsilon u)) \\ & = -\epsilon \operatorname{Re} j(T(\gamma)(v+\epsilon u))(T_0(\gamma)u). \end{aligned}$$

Since $T(\gamma)$ is continuous, $T(\gamma)(v+\epsilon u)$ converges to $T(\gamma)v$ as $\epsilon \to 0$. Because $T(\gamma)(v+\epsilon u) = T(\gamma)v + \epsilon T_0(\gamma)u$, we have

$$\begin{aligned} & \left\| \frac{T(\gamma)(v+\epsilon u)}{\|T(\gamma)(v+\epsilon u)\|} - \frac{T(\gamma)v}{\|T(\gamma)v\|} \right\| \\ = & \left\| \frac{T(\gamma)v + \epsilon T_0(\gamma)u}{\|T(\gamma)(v+\epsilon u)\|} - \frac{T(\gamma)v}{\|T(\gamma)(v+\epsilon u)\|} \frac{\|T(\gamma)(v+\epsilon u)\|}{\|T(\gamma)v\|} \right\| \\ = & \left\| \left(1 - \frac{\|T(\gamma)(v+\epsilon u)\|}{\|T(\gamma)v\|}\right) \frac{T(\gamma)v}{\|T(\gamma)(v+\epsilon u)\|} + \epsilon \frac{T_0(\gamma)u}{\|T(\gamma)(v+\epsilon u)\|} \right\|. \end{aligned}$$

Hence, by Proposition 1.3.5, $\operatorname{Re} j(T(\gamma)(v+\epsilon u))$ converges to $\operatorname{Re} j(T(\gamma)v)$ in B^* as $\epsilon \to 0$. On the other hand, for $\gamma \in K$ such that $T(\gamma)v = 0$, we have

$$\frac{\|T(\gamma)v\| - \|T(\gamma)(v+\epsilon u)\|}{\epsilon} = \frac{-\|\epsilon T_0(\gamma)u\|}{\epsilon} = -\|T_0(\gamma)u\|.$$

Hence we have

$$\lim_{\epsilon \to 0} \frac{F_{\alpha,p}(v) - F_{\alpha,p}(v+\epsilon u)}{\epsilon} = -\sum_{\gamma \in K} \left(\frac{\|T(\gamma)v\|^{p-1}}{F_{\alpha,p}(v)^{p-1}} \operatorname{Re} j(T(\gamma)v)(T_0(\gamma)u) \right) m(\gamma).$$

Therefore, as in the proof of Proposition 2.3.2, the proposition follows. □

Corollary 2.3.4. *Let $1 < p < \infty$ and α be an affine isometric action of Γ on $L^p(W,\nu)$, where (W,ν) is a measure space. For any $f \in L^p(W,\nu)$ such that $F_{\alpha,p}(f) > 0$, we have $|\nabla_- F_{\alpha,p}|(f) = 2\|G\|_{L^q(W,\nu)}/F_{\alpha,p}(f)^{p-1}$. Here q is the conjugate exponent of p, that is, $q = p/(p-1)$, and*

$$G(x) = \sum_{\gamma \in K} |f(x) - \alpha(\gamma)f(x)|^{p-2} \operatorname{Re}(f(x) - \alpha(\gamma)f(x))m(\gamma)$$

for $x \in W$, where $\operatorname{Re}(a)$ is the real part of a, and $|f(x) - \alpha(\gamma)f(x)|^{p-2} = 0$ if $f(x) = \alpha(\gamma)f(x)$ and $p < 2$.

Proof. For $f \in L^p(W,\nu)$, we have $j(f) = |f|^{p-2}\bar{f}/\|f\|^{p-1}_{L^p(W,\nu)}$, where $\bar{f}$ is the complex conjugation of f. Indeed, we have

$$\int_W \left(\frac{|f(x)|^{p-2}\bar{f}(x)}{\|f\|^{p-1}_{L^p(W,\nu)}} \right) f(x) d\nu(x) = \int_W \frac{|f(x)|^p}{\|f\|^{p-1}_{L^p(W,\nu)}} d\nu(x) = \frac{\|f\|^p_{L^p(W,\nu)}}{\|f\|^{p-1}_{L^p(W,\nu)}} = \|f\|_{L^p(W,\nu)}$$

and

$$\int_W \left| \frac{|f(x)|^{p-2}\bar{f}(x)}{\|f\|^{p-1}_{L^p(W,\nu)}} \right|^q d\nu(x) = \int_W \frac{|f(x)|^{(p-1)q}}{\|f\|^{(p-1)q}_{L^p(W,\nu)}} d\nu(x) = \int_W \frac{|f(x)|^p}{\|f\|^p_{L^p(W,\nu)}} d\nu(x) = 1.$$

We have thus proved the corollary. $\square$

Chapter 3

A fixed-point property for global Busemann NPC spaces

In this chapter, we will prove Theorem 1 and Theorem 2. These theorems say that the existence of a global fixed point of isometric action α on a global Busemann NPC space can be detected by the values of $|\nabla_{-}F_{\alpha,p}|$. These theorems generalize results in [IN05] and [IKN09] for isometric actions on Hadamard spaces. Also, we will give some examples of families satisfying the assumption of Theorem 2.

3.1 Proof of Theorem 1 and Theorem 2

Theorem 1. *Let G be a compactly generated group, (N,d) a global Busemann NPC space, and $1 \leq p \leq \infty$. For an isometric action α of G on (N,d), if there exists $C > 0$ such that $|\nabla_{-}F_{\alpha,p}|(x) \geq C$ for all $x \in N$ with $F_{\alpha,p}(x) > 0$, then α has a global fixed point.*

Proof. Since $F_{\alpha,p}$ is continuous and convex, $\inf_{x\in N} |\nabla_{-}F_{\alpha,p}|(x) = 0$ by Lemma 2.2.5. Hence, by the assumption of the theorem, there exists $x_0 \in N$ with $F_{\alpha,p}(x_0) = 0$. The point x_0 is a global fixed point of α. □

As mentioned in Introduction, in [IN05], to investigate the existence of a global fixed point of an isometric action α of a finitely generated group Γ on a Hadamard space Y, they used the energy functional E_α on the set Z of all α-equivariant maps from a countable Γ-space X equipped with an admissible weight into Y. If X is Γ, then Z can be identified with Y, and $E_\alpha(x) = \sum_{\gamma\in K} d(x,\alpha(\gamma)x)^2 m(\gamma)/2 = F_{\alpha,2}(x)^2/2$ for each $x \in Y$. To see the existence of a global fixed point of α, they assume that there exists $C > 0$ such that $|\nabla_{-}E_\alpha|(x)^2 \geq CE_\alpha(x)$ for all $x \in Y$. The inequality $|\nabla_{-}E_\alpha|(x)^2 \geq CE_\alpha(x)$ for all $x \in Y$ is equivalent to the inequality

$|\nabla_- F_{\alpha,2}|(x) \geq \sqrt{C/2}$ for all $x \in Y$ such that $F_{\alpha,p}(x) > 0$. Indeed,

$$\begin{aligned}\frac{E_\alpha(x) - E_\alpha(y)}{d(x,y)} &= \frac{(F_{\alpha,2}(x))^2 - (F_{\alpha,2}(y))^2}{2d(x,y)} \\ &= \frac{F_{\alpha,2}(x) - F_{\alpha,2}(y)}{2d(x,y)}(F_{\alpha,2}(x) + F_{\alpha,2}(y))\end{aligned}$$

for all $x, y \in Y$ with $x \neq y$. Hence we obtain

$$\begin{aligned}|\nabla_- E_\alpha|(x) &= \max\left\{\limsup_{y\to x,\, y\in M} \frac{E_\alpha(x) - E_\alpha(y)}{d(x,y)},\ 0\right\} \\ &= \max\left\{F_{\alpha,2}(x) \limsup_{y\to x,\, y\in M} \frac{F_{\alpha,2}(x) - F_{\alpha,2}(y)}{d(x,y)},\ 0\right\} \\ &= F_{\alpha,2}(x)|\nabla_- F_{\alpha,2}|(x)\end{aligned}$$

for all $x \in Y$. Hence their assumption is equivalent to the one in Theorem 1. Therefore Theorem 1 should be regarded as a generalization of a result in [IN05].

In the same spirit, using Propositions 2.3.2 and 2.3.3, for an affine isometric action α of Γ on a strictly convex, smooth, and real or uniformly convex and uniformly smooth Banach space B, the assumption that $|\nabla_- F_{\alpha,p}|(v) \geq C$ for all $v \in B$ with $F_{\alpha,p}(v) > 0$ can be replaced with the following:

$$\left\| \sum_{\gamma\in K} \|v - \alpha(\gamma)v\|^{p-1} m(\gamma) \operatorname{Re} j(v - \alpha(\gamma)v) \right\|_{B^*} \geq \frac{C}{2} F_{\alpha,p}(v)^{p-1}$$

for all $v \in B$.

Theorem 2. *Let Γ be a finitely generated group. Fix a finite generating subset K and a weight on K. Let $\mathcal{L}$ a family of global Busemann NPC spaces, and $1 \leq p \leq \infty$. Suppose $\mathcal{L}$ is stable under scaling ultralimit. Then the following are equivalent:*

(i) *For any $(N, d) \in \mathcal{L}$, every isometric action of Γ on (N, d) has a global fixed point.*

(ii) *For any $(N, d) \in \mathcal{L}$ and isometric action α of Γ on (N, d), there exists $C > 0$ such that $|\nabla_- F_{\alpha,p}|(x) \geq C$ for all $x \in N$ with $F_{\alpha,p}(x) > 0$.*

Furthermore, in (ii)*, C can be a constant independent of (N, d) and α.*

Proof. Because of Theorem 1, (ii) implies (i). We assume (i). To show that (ii) is true and that C is independent of (N, d) and α, we assume the contrary, and deduce a contradiction. By the assumption, we have $\{(N_n, d_n)\} \subset \mathcal{L}$, isometric actions α_n of Γ on (N_n, d_n), and $x_n \in N_n$ such that $F_{\alpha_n,p}(x_n) > 0$ and $|\nabla_- F_{\alpha_n,p}|(x_n) < 1/n$ for each n. Set

$$(N'_n, d'_n, x'_n) = (N_n, d_n/F_{\alpha_n,p}(x_n), x_n)$$

for each n. Then we can regard α_n as an isometric action of Γ on (N'_n, d'_n) for each n, and we denote it by α'_n. By the definitions of $F_{\alpha'_n,p}$ and $|\nabla_- F_{\alpha'_n,p}|$, we have $F_{\alpha'_n,p}(x'_n) = 1$ and $|\nabla_- F_{\alpha'_n,p}|(x'_n) = |\nabla_- F_{\alpha_n,p}|(x_n)$. Owing to the assumption, there exists $y'_n \in N'_n$ with $F_{\alpha'_n,p}(y'_n) = 0$ for each n. Therefore Corollary 2.2.3 implies that $|\nabla_- F_{\alpha'_n,p}|(x'_n) > 0$. Since $\{F_{\alpha'_n,p}(x'_n)\}$ is uniformly bounded independently of n, we can define α'_ω of $\{\alpha'_n\}$ and we have $F_{\alpha'_\omega,p}(x'_\omega) = 1$ by Lemma 2.1.2. We will show that x'_ω minimizes $F_{\alpha'_\omega,p}$. We take an arbitrary $y \in N_\omega$, and let $(y_n) \in N_\infty$ be a representative of y. Then there exists $\hat{C} > 0$ such that $\{d(x'_n, y_n)\}$ is bounded by $\hat{C}$ independently of n. By Proposition 2.2.2, we have

$$|\nabla_- F_{\alpha'_n,p}|(x'_n) \geq \frac{F_{\alpha'_n,p}(x'_n) - F_{\alpha'_n,p}(y_n)}{d_n(x'_n, y_n)}$$

for each n. By the assumption, we get

$$F_{\alpha'_n,p}(x'_n) - F_{\alpha'_n,p}(y_n) \leq d_n(x'_n, y_n)|\nabla_- F_{\alpha'_n,p}|(x'_n) < \frac{\hat{C}}{n}.$$

By Lemma 2.1.2, we have

$$F_{\alpha'_\omega,p}(x'_\omega) - F_{\alpha'_\omega,p}(y) = \omega\text{-}\lim_n (F_{\alpha'_n,p}(x'_n) - F_{\alpha'_n,p}(y_n)) \leq 0.$$

Therefore $F_{\alpha'_\omega,p}(x'_\omega) \leq F_{\alpha'_\omega,p}(y)$ for all $y \in N_\omega$. This means that x'_ω minimizes $F_{\alpha'_\omega,p}$. However, x'_ω is not a global fixed point, because $F_{\alpha'_\omega,p}(x'_\omega) = 1$. This contradicts (i). $\square$

3.2 Examples

Next, we give some examples of a family of global Busemann NPC spaces which is stable under scaling ultralimit.

For a fixed p with $1 < p < \infty$, the family of all L^p is an example of such a family (see [AK90, II Theorem 2.9] and [Hei80]). In particular, the family of all Hilbert spaces is also an example of such a family. The following are also examples of such a family.

Example 3.2.1. Let $\delta : (0, 2] \to (0, 1]$ be a left continuous, monotone increasing function. Let $\mathcal{L}_\delta$ be the family consisting of all Banach spaces B such that

$\delta_B(\epsilon) \geq \delta(\epsilon)$. Then $\mathcal{L}_\delta$ is a family of global Busemann NPC spaces which is stable under scaling ultralimit, as we explain below. First note that $B \in \mathcal{L}_\delta$ is uniformly convex, hence B is a global Busemann NPC space. Let ω be a non-principal ultrafilter, $\{(B_n, \| \ \|_n, o_n)\} \subset \mathcal{L}_\delta$ and $\{r_n\} \subset \mathbb{R}$ with $r_n > 0$. Set $(B_\omega, \| \ \|'_\omega) := \omega\text{-}\lim_n(B_n, r_n\| \ \|_n, o_n)$. For $0 < \epsilon \leq 2$, we take $u, v \in B_\omega$ with $\|u\|'_\omega = \|v\|'_\omega = 1$ and $\|u - v\|'_\omega \geq \epsilon$. For representatives (u_n) of u and (v_n) of v and $0 < \eta < \epsilon$, if $|r_n\|u_n - v_n\|_n - \|u - v\|'_\omega| < \eta$, then we have $r_n\|u_n - v_n\|_n > \|u - v\|'_\omega - \eta \geq \epsilon - \eta$. Hence

$$\{n \in \mathbb{N} : | \ r_n\|u_n\|_n - 1| < \eta, \ | \ r_n\|v_n\|_n - 1| < \eta, \ r_n\|u_n - v_n\|_n > \epsilon - \eta\} \in \omega.$$

This set is contained in

$$\left\{n \in \mathbb{N} : \frac{r_n\|u_n\|_n}{1+\eta} < 1, \ \frac{r_n\|v_n\|_n}{1+\eta} < 1, \ \frac{r_n\|u_n - v_n\|_n}{1+\eta} > \frac{\epsilon - \eta}{1+\eta}\right\}.$$

Therefore, by the uniform convexity and the assumption on the modulus of convexity,

$$\left\{n \in \mathbb{N} : \frac{r_n\|u_n + v_n\|_n}{2(1+\eta)} \leq 1 - \delta\left(\frac{\epsilon - \eta}{1+\eta}\right)\right\} \in \omega.$$

Since η is arbitrary, we have

$$\frac{\|u + v\|'_\omega}{2} \leq \inf_{0<\eta<\epsilon}(1+\eta)\left(1 - \delta\left(\frac{\epsilon - \eta}{1+\eta}\right)\right) = 1 - \delta(\epsilon),$$

that is, $\delta_{B_\omega}(\epsilon) \geq \delta(\epsilon)$ for $0 < \epsilon \leq 2$.

Example 3.2.2. We fix $k > 0$. Let $\mathcal{L}_k$ be the family of all global Busemann NPC spaces (N, rd), where $r > 0$ and (N, d) is a global Busemann NPC space with the following condition: For any $x \in N$ and shortest geodesic $c : [0, l] \to N$,

$$kd(x, c(tl))^2 \leq (1-t)d(x, c(0))^2 + td(x, c(l))^2 - (1-t)td(c(0), c(l))^2$$

for $0 \leq t \leq 1$. Then $\mathcal{L}_k$ is stable under scaling ultralimit. In particular, $\mathcal{L}_1$ consists of all Hadamard spaces.

The reason why $\mathcal{L}_k$ is stable under scaling ultralimit is the following: Let ω be a non-principal ultrafilter. By the definition of $\mathcal{L}_k$, we need only to show that for $\{(N_n, d_n, o_n)\} \subset \mathcal{L}_k$, $(N_\omega, d_\omega) = \omega\text{-}\lim_n(N_n, d_n, o_n) \in \mathcal{L}_k$. Take arbitrary $x, y \in N_\omega$ and their representatives $(x_n), (y_n) \in N_\infty$ respectively. For each n, we take a shortest geodesic $c_n : [0, l_n] \to N_n$ joining x_n to y_n, then $c_\omega : [0, l_\omega] \to N_\omega$ defined by $c_\omega(t) := \omega\text{-}\lim_n c_n(tl_n/l_\omega)$ is a shortest geodesic joining x to y, and $l_\omega = d_\omega(x, y)$. Assume there exists another shortest geodesic $\tilde{c} : [0, l] \to N_\omega$

joining x to y, where $l = l_\omega$. Let $(\tilde{c}_n(t))$ be an arbitrary representative of $\tilde{c}(t)$ for each $t \in [0, l]$. By the definition of $\mathcal{L}_k$, for each n we have

$$\begin{aligned}
& kd_n(\tilde{c}_n(tl), c_n(tl_n))^2 \\
\leq\ & (1-t)d_n(\tilde{c}_n(tl), c_n(0))^2 + td_n(\tilde{c}_n(tl), c_n(l_n))^2 - (1-t)td_n(c_n(0), c_n(l_n))^2 \\
=\ & (1-t)\{d_n(\tilde{c}_n(tl), c_n(0))^2 - (td_n(c_n(0), c_n(l_n)))^2\} \\
& \quad + t\{d_n(\tilde{c}_n(tl), c_n(l_n))^2 - ((1-t)d_n(c_n(0), c_n(l_n)))^2\} \\
=\ & (1-t)\{d_n(\tilde{c}_n(tl), x_n)^2 - (tl_n)^2\} + t\{d_n(\tilde{c}_n(tl), y_n)^2 - ((1-t)l_n)^2\}
\end{aligned}$$

for $0 \leq t \leq 1$. Hence we obtain

$$kd_\omega(\tilde{c}(tl), c_\omega(tl_\omega))^2 \leq (1-t)\{(tl)^2 - (tl_\omega)^2\} + t\{((1-t)l)^2 - ((1-t)l_\omega)^2\} = 0$$

for $0 \leq t \leq 1$. This contradicts the assumption that $\tilde{c} \neq c_\omega$. Therefore the shortest geodesic joining x to y is unique, and it is the ultralimit of shortest geodesics. Hence, by Lemma 1.2.2 and Lemma 1.2.3, the Busemann NPC inequalities for (N_n, d_n) implies the Busemann NPC inequality for (N_ω, d_ω). By Lemma 1.2.2, Lemma 1.2.3 and Lemma 1.2.4, for any $x \in N_\omega$ and shortest geodesic $c : [0, l] \to N_\omega$, we have

$$kd_\omega(x, c(t))^2 \leq (1-t)d_\omega(x, c(0))^2 + td_\omega(x, c(l))^2 - (1-t)td_\omega(c(0), c(l))^2$$

for $0 \leq t \leq 1$.

Example 3.2.3. We fix $k > 0$. Let $\mathcal{L}'_k$ be a family of all global Busemann NPC spaces (N, rd), where $r > 0$ and (N, d) is a global Busemann NPC space with the following condition: For any $x \in N$ and shortest geodesic $c : [0, l] \to N$,

$$d(x, c(t))^2 \leq (1-t)d(x, c(0))^2 + td(x, c(l))^2 - k(1-t)td(c(0), c(l))^2$$

for $0 \leq t \leq 1$. The metric space satisfying this inequality was introduced by Ohta [Oht07]. He proves that this inequality implies the inequality in the definition of $\mathcal{L}_k$. For an ultralimit (N_ω, d_ω) of a sequence of global Busemann NPC spaces in $\mathcal{L}'_k$, using the proof that $\mathcal{L}_k$ is stable under scaling ultralimit, we can show that a shortest geodesic joining arbitrary two points in N_ω is unique, and it is the ultralimit of shortest geodesics. Hence, by Lemma 1.2.2 and Lemma 1.2.3, the Busemann NPC inequality for (N_ω, d_ω) holds. By Lemma 1.2.2, Lemma 1.2.3 and Lemma 1.2.4, for any $x \in N_\omega$ and shortest geodesic $c : [0, l] \to N_\omega$, we have

$$d_\omega(x, c(t))^2 \leq (1-t)d_\omega(x, c(0))^2 + td_\omega(x, c(l))^2 - k(1-t)td_\omega(c(0), c(l))^2$$

for $0 \leq t \leq 1$. Therefore, $\mathcal{L}'_k$ is also stable under scaling ultralimit.

Chapter 4

A fixed-point property for Banach spaces

In this chapter, first, we will give a proof of Theorem 4. Second, we will apply Theorem 4 to the left regular representation $\lambda_{\Gamma,p}$ of a finitely generated group Γ on $\ell^p(\Gamma)$ with $1 < p < \infty$. Finally, we will prove Theorem 3 and Proposition 5, which state relations between Property (F_B), Property (T_B) and the values of $|\nabla_- F_{\alpha,p}|$.

4.1 Proof of Theorem 4

Let Γ be a finitely generated group, K a finite generating subset of Γ, m a weight on K, and B a strictly convex Banach space.

Theorem 4. *Let π be a linear isometric representation of Γ on B, and $1 \leq p \leq \infty$. Suppose π has no non-trivial invariant vector. Then the following are equivalent:*

(i) *The first cohomology $H^1(\Gamma, \pi)$ vanishes.*

(ii) *For any affine isometric action α of Γ on B with the linear part π, there exists $C > 0$ such that $|\nabla_- F_{\alpha,p}|(v) \geq C$ for all $v \in B$ with $F_{\alpha,p}(v) > 0$.*

Furthermore, in (ii)*, C can be a constant independent of α.*

Proof. Due to Theorem 1, for an affine isometric action α, if there exists $C > 0$ such that $|\nabla_- F_{\alpha,p}|(v) \geq C$ for all $v \in B$ such that $F_{\alpha,p}(v) > 0$, then α has a global fixed point. Hence $H^1(\Gamma, \pi)$ vanishes.

Conversely, we assume that $H^1(\Gamma, \pi)$ vanishes. Hence $B^1(\pi)$ coincides with $Z^1(\pi)$. Since π has no non-trivial invariant vector, $d : B \to B^1(\pi)$ is one-to-one. Hence the open mapping theorem implies that the inverse map d^{-1} of d

is bounded. Thus there exists $C_p > 0$ satisfying $\|v\| = \|d^{-1}(dv)\| \leq C_p\|dv\|_p$ for all $v \in B$. Take an arbitrary affine isometric action α of Γ on B with the linear part π. Then there exists a global fixed point $v_0 \in B$ of α. Since $\pi(\gamma)v = \alpha(\gamma)(v+v_0) - v_0$ for all $v \in B$ and $\gamma \in \Gamma$, we have $F_{\alpha,p}(v+v_0) = F_{\pi,p}(v)$ for all $v \in B$. Therefore we may assume that α coincides with π. By the definition of d, we have $F_{\pi,p}(v) = \|dv\|_p$ for all $v \in B$. Hence we have

$$|\nabla_{-}F_{\pi,p}|(v) \geq \lim_{\epsilon \to 0} \frac{F_{\pi,p}(v) - F_{\pi,p}(v_\epsilon)}{\|v - v_\epsilon\|} = \lim_{\epsilon \to 0} \frac{\|dv\|_p - \|dv_\epsilon\|_p}{\epsilon\|v\|} = \lim_{\epsilon \to 0} \frac{\epsilon\|dv\|_p}{\epsilon\|v\|} \geq \frac{1}{C_p}$$

for all non-trivial $v \in B$, where $v_\epsilon := (1-\epsilon)v$ for $\epsilon > 0$. Since $F_{\pi,p}(0) = 0$, we have completed the proof. $\square$

Suppose that K is symmetric and m is symmetric in the rest of this section. Let $\alpha = \pi + c$ be an affine isometric action of Γ on B. Then we have $v - \alpha(\gamma)v = (dv - c)(\gamma)$ for all $\gamma \in \Gamma$ and $v \in B$. By Proposition 2.3.2 and Proposition 2.3.3, if B is strictly convex, smooth and real, or uniformly convex and uniformly smooth, then for $1 < p < \infty$

$$|\nabla_{-}F_{\alpha,p}|(v) = \frac{2}{\|dv - c\|_p^{p-1}} \left\| \sum_{\gamma \in K} \|(dv-c)(\gamma)\|^{p-1} m(\gamma) \operatorname{Re} j((dv-c)(\gamma)) \right\|_{B^*}$$

for all $v \in B$ such that $\|dv - c\|_p > 0$. Here Re is trivial, when B is real. Hence, for $C > 0$, $|\nabla_{-}F_{\alpha,p}|(v) \geq C$ for all $v \in B$ such that $F_{\alpha,p}(v) > 0$ if and only if

$$\left\| \sum_{\gamma \in K} \|(dv-c)(\gamma)\|^{p-1} m(\gamma) \operatorname{Re} j((dv-c)(\gamma)) \right\|_{B^*} \geq \frac{C}{2}\|dv - c\|_p^{p-1}$$

for all $v \in B$. There exists a one-to-one correspondence between $Z^1(\pi)$ and the set of all affine isometric actions with the linear part π when the origin of B is fixed. Since $dv - c$ is a π-cocycle, from Theorem 4, we have the following corollary.

Corollary 4.1.1. *Let π be a linear isometric representation of Γ on B, and $1 < p < \infty$. Suppose that B is either strictly convex, smooth and real, or uniformly convex and uniformly smooth, and π has no non-trivial invariant vector. Then $H^1(\Gamma, \pi)$ vanishes if and only if there exists $C > 0$ such that*

$$\left\| \sum_{\gamma \in K} \|c(\gamma)\|^{p-1} m(\gamma) \operatorname{Re} j(c(\gamma)) \right\|_{B^*} \geq C\|c\|_p^{p-1}$$

for all $c \in Z^1(\pi)$.

4.2 p-Laplacian

Let Γ be a finitely generated infinite group, K a symmetric finite generating subset of Γ, m a symmetric weight on K, and $1 < p < \infty$. We denote by $\mathbb{C}$ the space of all complex numbers.

We denote by $\mathcal{F}(\Gamma)$ the space of all complex-valued (or real-valued) functions on Γ. The *left regular representation* λ_Γ of Γ on $\mathcal{F}(\Gamma)$ is defined by $\lambda_\Gamma(\gamma)f(\gamma') = f(\gamma^{-1}\gamma')$ for each $f \in \mathcal{F}(\Gamma)$ and $\gamma, \gamma' \in \Gamma$. The Lebesgue space $\ell^p(\Gamma)$ is the Banach space $\{f \in \mathcal{F}(\Gamma) : \sum_{\gamma\in\Gamma} |f(\gamma)|^p < \infty\}$ with the norm $\|f\|_{\ell^p(\Gamma)} := (\sum_{\gamma\in\Gamma} |f(\gamma)|^p)^{1/p}$. The restriction of λ_Γ to $\ell^p(\Gamma)$ is a linear isometric representation with no non-trivial invariant vector, and we denote it by $\lambda_{\Gamma,p}$. We define a linear map d on $\mathcal{F}(\Gamma)$ into itself by $df(\gamma) := f - \lambda_\Gamma(\gamma)f$ for each $f \in \mathcal{F}(\Gamma)$ and $\gamma \in \Gamma$. We say that $f \in \mathcal{F}(\Gamma)$ is *p-Dirichlet finite* if $df \in \ell^p(\Gamma)$, and we denote by $D_p(\Gamma)$ the space of all p-Dirichlet finite functions.

The space of all constant functions on Γ is a subspace of $D_p(\Gamma)$, and is regarded as $\mathbb{C}$ (or $\mathbb{R}$). Since this is the kernel of d, we can define a norm on $D_p(\Gamma)/\mathbb{C}$ (or $D_p(\Gamma)/\mathbb{R}$) by $\|f\|_{D_p(\Gamma)} = (\sum_{\gamma\in K} \|df(\gamma)\|^p_{\ell^p(\Gamma)} m(\gamma))^{1/p}$. The space $\ell^p(\Gamma)$ is also a subspace of $D_p(\Gamma)$. Because

$$\begin{aligned}
\lambda_{\Gamma,p}(\gamma)df(\gamma') + df(\gamma) &= \lambda_\Gamma(\gamma)f - \lambda_\Gamma(\gamma)\lambda_\Gamma(\gamma')f + f - \lambda_\Gamma(\gamma)f \\
&= f - \lambda_\Gamma(\gamma\gamma')f \\
&= df(\gamma\gamma')
\end{aligned}$$

for all $f \in D_p(\Gamma)$ and $\gamma, \gamma' \in \Gamma$, we obtain $df \in Z^1(\lambda_{\Gamma,p})$ for $f \in D_p(\Gamma)$.

Furthermore, it is known that $d(D_p(\Gamma)) = Z^1(\lambda_{\Gamma,p})$. Recall that $B^1(\lambda_{\Gamma,p}) = d(\ell^p(\Gamma))$. Therefore d induces the isometric isomorphism from $D_p(\Gamma)/\mathbb{C}$ (or $D_p(\Gamma)/\mathbb{R}$) onto $Z^1(\Gamma)$ and the (linear) isomorphism from $D_p(\Gamma)/(\ell^p(\Gamma) \oplus \mathbb{C})$ (or $D_p(\Gamma)/(\ell^p(\Gamma) \oplus \mathbb{R})$) onto $H^1(\Gamma, \lambda_\Gamma)$. This is pointed out in [Pul03] and [Pul06]. Hence, for any affine isometric action α on $\ell^p(\Gamma)$ with the linear part $\lambda_{\Gamma,p}$, there exists a unique $f_\alpha \in D_p(\Gamma)$ up to constant such that the cocycle part c of α coincides with df_α and $\|c\|_p = \|f_\alpha\|_{D_p(\Gamma)}$.

The p-Laplacian $\Delta_p f$ of $f \in D_p(\Gamma)$ is defined by

$$\Delta_p f(x) := \sum_{\gamma\in K} |df(\gamma)(x)|^{p-2}\,\mathrm{Re}(df(\gamma)(x))m(\gamma).$$

Since $F_{\alpha,p}(f) = \|df - df_\alpha\|_p = \|f - f_\alpha\|_{D_p(\Gamma)}$ for all $f \in \ell^p(\Gamma)$, by Corollary 2.3.4, we have

$$|\nabla_- F_{\alpha,p}|(f) = \frac{2\|\Delta_p(f - f_\alpha)\|_{\ell^q(\Gamma)}}{\|f - f_\alpha\|^{p-1}_{D_p(\Gamma)}}$$

for all $f \in \ell^p(\Gamma)$ such that $F_{\alpha,p}(f) > 0$. Using Corollary 4.1.1, we have the following corollary of Theorem 4.

Corollary 4.2.1. *The first cohomology $H^1(\Gamma, \lambda_{\Gamma,p})$ vanishes if and only if there exists $C > 0$ such that $\|\Delta_p f\|_{\ell^q(\Gamma)}^q \geq C\|f\|_{D_p(\Gamma)}^{p-1}$ for all $f \in D_p(\Gamma)$.*

4.3 Properties (F_B), (T_B) and $(AG_{B,p})$

In this section, we review the definitions of Property (F_B) and Property (T_B), which ware introduced in [BFGM07], and will introduce a new property: Property $(AG_{B,p})$. Also, we will prove Theorem 3 and Proposition 5.

Let Γ be a finitely generated group, K a finite generating subset, and m a weight. Let B be a Banach space and $1 < p < \infty$.

Definition 4.3.1 ([BFGM07]). We say Γ has *Property* (F_B) if every affine isometric action of Γ on B has a global fixed point.

For a linear isometric representation π of Γ on B, we denote by $B^{\pi(\Gamma)}$ the closed subspace consisting of all invariant vectors of π. We can define a linear isometric representation π' of Γ on $B/B^{\pi(\Gamma)}$ by $\pi'(\gamma)[v] := [\pi(\gamma)v]$ for each $v \in B$ and $\gamma \in \Gamma$, where $[v]$ is the equivalence class of v.

Definition 4.3.2 ([BFGM07]). We say Γ has *Property* (T_B) if, for every nontrivial linear isometric representation π of Γ on B, there exists $C > 0$ such that $\max_{\gamma\in K} \|u - \pi'(\gamma)u\| \geq C\|u\|$ for all $u \in B/B^{\pi(\Gamma)}$.

We rewrite the theorem due to A. Guichardet.

Theorem 4.3.3 ([Gui72]). *If Γ has Property (F_B), then it has Property (T_B).*

For an affine isometric action $\alpha = \pi + c$ of Γ on B, we can define an affine isometric action α' on $B' := B/B^{\pi(\Gamma)}$ by $\alpha'(\gamma)[v] := [\alpha(\gamma)v] = \pi'(\gamma)[v] + [c(\gamma)]$ for each $v \in B$ and $\gamma \in \Gamma$.

Definition 4.3.4. We say Γ has *Property* $(AG_{B,p})$ if, for every affine isometric action α of Γ on B, there exists $C > 0$ such that $|\nabla_- F_{\alpha',p}|(u) \geq C$ for all $u \in B'$ with $F_{\alpha',p}(u) > 0$.

Theorem 3. *If Γ has Property (F_B), then it has Property $(AG_{B,p})$ for $1 < p < \infty$.*

Proof. Take an arbitrary affine isometric action $\alpha = \pi + c$ of Γ on B. Since Γ has Property (F_B), there exists a global fixed point v_0 of α. We can regard $[v_0]$ as the origin 0 of B', hence we may assume that α' coincides with π'. By Guichardet's theorem, we have $C > 0$ such that $\max_{\gamma \in K} \|u - \pi'(\gamma)u\| \geq C\|u\|$ for all $u \in B'$. For $v \in B'$, since

$$F_{\pi',p}(v)^p = \sum_{\gamma \in K} \|v - \pi'(\gamma)v\|^p m(\gamma) \geq \max_{\gamma \in K} \|v - \pi'(\gamma)v\|^p \min_{\gamma \in K} m(\gamma),$$

we obtain $F_{\pi',p}(v) \geq C\|v\|(\min_{\gamma \in K} m(\gamma))^{1/p}$. Set $v_\epsilon = (1-\epsilon)v$ for $\epsilon > 0$. If $v \neq 0$, because $F_{\pi',p}(v_\epsilon) = (1-\epsilon)F_{\pi',p}(v)$ for $\epsilon > 0$, we have

$$|\nabla_- F_{\pi',p}|(v) \geq \lim_{\epsilon \to 0} \frac{F_{\pi',p}(v) - F_{\pi',p}(v_\epsilon)}{\|v - v_\epsilon\|} = \lim_{\epsilon \to 0} \frac{\epsilon F_{\pi',p}(v)}{\epsilon \|v\|} \geq C \left(\min_{\gamma \in K} m(\gamma) \right)^{1/p}.$$

Since $F_{\pi',p}(0) = 0$, this completes the proof. $\square$

We can not use Theorem 4 to prove Theorem 3, because some affine isometric action on B' may not extend to B.

The converse of Theorem 3 is false, because $\mathbb{Z}$ has Property $(AG_{\mathbb{R},p})$ with $1 < p < \infty$ but does not have Property $(F_{\mathbb{R}})$. Indeed, $\mathbb{Z}$ acts on $\mathbb{R}$ by isometric translations: $\alpha(n)t := t + n$ for each $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Hence $\mathbb{Z}$ does not have Property $(F_{\mathbb{R}})$. On the other hand, for an affine isometric action α with the trivial linear part, $F_{\alpha',p} = 0$. For an affine isometric action α with a non-trivial linear part, rescaling the metric of $\mathbb{R}$, we can describe α' as $\alpha'(2n+1)t := -t + s$ and $\alpha'(2n)t := t$ for each $t \in \mathbb{R}$ and $n \in \mathbb{Z}$, where $s \in \mathbb{R}$. The action α' fixes only $s/2$. The set $K = \{1\}$ is a finite generating subset of $\mathbb{Z}$ and a function m defined by $m(1) = 1$ is a weight. Then $F_{\alpha',p}(t) = |2t - s|$ for all $t \in \mathbb{R}$. Hence $|\nabla_- F_{\alpha',p}|(t) = 2 > 0$ unless $t = s/2$. Therefore $\mathbb{Z}$ has Property $(AG_{\mathbb{R},p})$.

Proposition 5. *Suppose that Γ is Abelian, K is symmetric, m is symmetric and B is uniformly convex, uniformly smooth, and real. If Γ has Property (T_B), then it has Property $(AG_{B,p})$ for all $1 < p < \infty$.*

Proof. Let $\alpha = \pi + c$ be an arbitrary affine isometric action of Γ on B with a non-trivial linear part π. Note that B' is also a uniformly convex and uniformly smooth real Banach space. Let $u \in B'$ such that $F_{\alpha',p}(u) > 0$. Set

$$\begin{aligned} T(\gamma)u &= u - \alpha'(\gamma)u, & T_0(\gamma)u &= u - \pi'(\gamma)u, \\ O &= \sum_{\gamma \in K} m(\gamma)[c(\gamma)], & X(u) &= \sum_{\gamma \in K} m(\gamma)\pi'(\gamma)u, \\ Y(u) &= \sum_{\gamma \in K} m(\gamma)\alpha'(\gamma)u, & W(u) &= u - Y(u) = u - X(u) - O \end{aligned}$$

for each $\gamma \in K$. By Proposition 2.3.2, we have

$$|\nabla_{-}F_{\alpha',p}|(u) \geq \frac{1}{F_{\alpha',p}(u)^{p-1}} \sum_{\gamma \in K} \|T(\gamma)u\|^{p-1} \frac{j(T(\gamma)u)(T_0(\gamma)w)}{\|w\|} m(\gamma)$$

for all non-trivial $w \in B'$. For any $\gamma \in K$

$$\begin{aligned}
j(T(\gamma)u)(T_0(\gamma)W(u)) &= j(T(\gamma)u)(T_0(\gamma)u - T_0(\gamma)Y(u)) \\
&= j(T(\gamma)u)(T(\gamma)u - T(\gamma)Y(u)) \\
&= \|T(\gamma)u\| - j(T(\gamma)u)T(\gamma)Y(u) \\
&\geq \|T(\gamma)u\| - \{\|T(\gamma)u + T(\gamma)Y(u)\| - \|T(\gamma)u\|\} \\
&= 2\|T(\gamma)u\| - \|T(\gamma)u + T(\gamma)Y(u)\|.
\end{aligned}$$

Because $\sum_{\kappa \in K} m(\kappa) = 1$, we have

$$\begin{aligned}
& \|T(\gamma)u + T(\gamma)Y(u)\| \\
= & \left\| \sum_{\kappa \in K} m(\kappa)T(\gamma)u + T(\gamma)\left(\sum_{\kappa \in K} m(\kappa)\alpha'(\kappa)u\right) \right\| \\
= & \left\| \sum_{\kappa \in K} m(\kappa)T(\gamma)u + T_0(\gamma)\left(\sum_{\kappa \in K} m(\kappa)\alpha'(\kappa)u\right) + \sum_{\kappa \in K} m(\kappa)[c(\gamma)] \right\| \\
= & \left\| \sum_{\kappa \in K} m(\kappa)\left(T(\gamma)u + T(\gamma)\alpha'(\kappa)u\right) \right\| \\
\leq & \sum_{\kappa \in K} m(\kappa)\|T(\gamma)u + T(\gamma)\alpha'(\kappa)u\|
\end{aligned}$$

for all $\gamma \in K$. Since Γ is Abelian, we have

$$\begin{aligned}
T(\gamma)\alpha'(\kappa)u &= \alpha'(\kappa)u - \alpha'(\gamma)\alpha'(\kappa)u \\
&= \alpha'(\kappa)u - \alpha'(\kappa)\alpha'(\gamma)u \\
&= \pi'(\kappa)u - \pi'(\kappa)\alpha'(\gamma)u \\
&= \pi'(\kappa)(T(\gamma)u)
\end{aligned}$$

for all $\gamma, \kappa \in K$. Hence we get

$$\|T(\gamma)u + T(\gamma)\alpha'(\kappa)u\| = \|T(\gamma)u + \pi'(\kappa)(T(\gamma)u)\|$$

for all $\gamma, \kappa \in K$. Due to Property (T_B), there exists $C(\pi') > 0$ such that any $\gamma \in K$ satisfies

$$\|T(\gamma)u - \pi'(\kappa_\gamma)(T(\gamma)u)\| \geq C(\pi')\|T(\gamma)u\|$$

for some $\kappa_\gamma \in K$. Since B' is uniformly convex, we have

$$1 - \frac{1}{2}\left\| \frac{T(\gamma)u}{\|T(\gamma)u\|} + \pi'(\kappa_\gamma)\frac{T(\gamma)u}{\|T(\gamma)u\|} \right\| \geq \delta_{B'}(C(\pi'))$$

for all $\gamma \in K$ with $T(\gamma)u \neq 0$. Then we have

$$2\|T(\gamma)u\| - \|T(\gamma)u + \pi'(\kappa_\gamma)T(\gamma)u\| \geq 2\delta_{B'}(C(\pi'))\|T(\gamma)u\|$$

for all $\gamma \in K$. On the other hand, by Hölder's inequality,

$$\|W(u)\| \leq \sum_{\gamma\in K} m(\gamma)\|T(\gamma)u\| \leq \left(\sum_{\gamma\in K} m(\gamma)\|T(\gamma)u\|^p\right)^{1/p} = F_{\alpha',p}(u).$$

Therefore, in the case that $\|W(u)\| \neq 0$, we have

$$\begin{aligned}
& \sum_{\gamma\in K} \|T(\gamma)u\|^{p-1}\frac{j(T(\gamma)u)(T_0(\gamma)W(u))}{\|W(u)\|}m(\gamma) \\
\geq\ & \sum_{\gamma\in K} \|T(\gamma)u\|^{p-1}\frac{\sum_{\kappa\in K} m(\kappa)(2\|T(\gamma)u\| - \|T(\gamma)u + \pi'(\kappa)(T(\gamma)u)\|)}{\|W(u)\|}m(\gamma) \\
\geq\ & \sum_{\gamma\in K} \|T(\gamma)u\|^{p-1}\frac{m(\kappa_\gamma)(2\|T(\gamma)u\| - \|T(\gamma)u + \pi'(\kappa)(T(\gamma)u)\|)}{\|W(u)\|}m(\gamma) \\
\geq\ & \sum_{\gamma\in K} \|T(\gamma)u\|^{p-1}\frac{\min_{\kappa\in K} m(\kappa)2\delta_{B'}(C(\pi'))\|T(\gamma)u\|}{F_{\alpha',p}(u)}m(\gamma) \\
=\ & \min_{\kappa\in K} m(\kappa)2\delta_{B'}(C(\pi'))F_{\alpha',p}(u)^{p-1},
\end{aligned}$$

that is, $C := 2\delta_{B'}(C(\pi'))\min_{\kappa\in K} m(\kappa) \leq |\nabla_- F_{\alpha',p}|(u)$.

In the case that $\|W(u)\| = 0$, if $O = u - X(u) \neq 0$, we have

$$W(au) = au - X(au) - O = a(u - X(u)) - O = (a-1)O$$

for $a \in \mathbb{R}$. Hence $W(au) \neq 0$ if $a \neq 1$. Because $T(\gamma)(au) = T(\gamma)u - (1-a)T_0(\gamma)u$, we have

$$\begin{aligned}
& \left\| \frac{T(\gamma)(au)}{\|T(\gamma)(au)\|} - \frac{T(\gamma)u}{\|T(\gamma)u\|} \right\| \\
=\ & \left\| \frac{T(\gamma)u - (1-a)T_0(\gamma)u}{\|T(\gamma)(au)\|} - \frac{T(\gamma)u}{\|T(\gamma)(au)\|}\frac{\|T(\gamma)(au)\|}{\|T(\gamma)u\|} \right\| \\
=\ & \left\| \left(1 - \frac{\|T(\gamma)(au)\|}{\|T(\gamma)u\|}\right)\frac{T(\gamma)u}{\|T(\gamma)(au)\|} - (1-a)\frac{T_0(\gamma)u}{\|T(\gamma)(au)\|} \right\|
\end{aligned}$$

for all $\gamma \in K$ with $\|T(\gamma)(u)\| \neq 0$ and $a < 1$ which is sufficiently close to 1. Hence, by Proposition 1.3.5, $j(T(\gamma)(au))$ converges to $j(T(\gamma)u)$ in B^* as $a \to 1$ for all $\gamma \in K$ with $\|T(\gamma)u\| \neq 0$. For $\gamma \in K$ with $\|T(\gamma)u\| = 0$, we have

$$\lim_{a \nearrow 1} |\|T(\gamma)(au)\|^{p-1} j(T(\gamma)(au))w| \leq \lim_{a \nearrow 1} \|T(\gamma)(au)\|^{p-1} \|w\| = 0$$

for all $w \in B$. Thus, using the conclusion in the case that $W(u) \neq 0$ and the proof of Proposition 2.3.2, we have

$$\begin{aligned} C &\leq \lim_{a \nearrow 1} \sum_{\gamma \in K} \|T(\gamma)(au)\|^{p-1} \frac{j(T(\gamma)(au))(T_0(\gamma)W(au))}{\|W(au)\|} m(\gamma) \\ &= \lim_{a \nearrow 1} 2 \sum_{\gamma \in K} \|T(\gamma)(au)\|^{p-1} j(T(\gamma)(au)) \frac{W(au)}{\|W(au)\|} m(\gamma) \\ &= 2 \sum_{\gamma \in K} \|T(\gamma)u\|^{p-1} j(T(\gamma)u) \frac{-O}{\|O\|} m(\gamma) \\ &\leq |\nabla_{-} F_{\alpha', p}|(u). \end{aligned}$$

On the other hand, if $O = 0$, u must be trivial. We prove this by contradiction. Suppose u is non-trivial. By Property (T_B), there exist $C(\pi') > 0$ and $\gamma_1 \in K$ such that $\|\pi'(\gamma_1)u - u\| \geq C(\pi')\|u\|$. Since B' is uniformly convex, we have

$$1 - \frac{\|\pi'(\gamma_1)u + u\|}{2\|u\|} \geq \delta_{B'}(C(\pi')).$$

Since $W(u)$ and O are trivial, we have $u = X(u)$. Hence we obtain

$$\begin{aligned} \|2u\| &= \|u + X(u)\| \\ &= \left\| \sum_{\gamma \in K} m(\gamma)(u + \pi'(\gamma)u) \right\| \\ &\leq \sum_{\gamma \in K} m(\gamma) \|u + \pi'(\gamma)u\| \\ &\leq m(\gamma_1)(2 - 2\delta_{B'}(C(\pi')))\|u\| + \sum_{\gamma \in K; \gamma \neq \gamma_1} m(\gamma)\|u + \pi'(\gamma)u\| \\ &\leq 2\|u\| - 2\delta_{B'}(C(\pi'))m(\gamma_1)\|u\| \\ &< \|2u\|. \end{aligned}$$

This is a contradiction. Hence we conclude that u is trivial, and $w \in B' \backslash \{u\}$ satisfies $W(\epsilon w) \neq 0$ for all $\epsilon > 0$. Because $T(\gamma)(\epsilon w) = \epsilon T(\gamma)w - (1-\epsilon)[c(\gamma)]$ and

$T(\gamma)u = -[c(\gamma)]$, we have

$$
\begin{aligned}
& \left\| \frac{T(\gamma)(\epsilon w)}{\|T(\gamma)(\epsilon w)\|} - \frac{T(\gamma)u}{\|T(\gamma)u\|} \right\| \\
= & \left\| \frac{\epsilon T(\gamma)w - (1-\epsilon)[c(\gamma)]}{\|T(\gamma)(\epsilon w)\|} - \frac{T(\gamma)u}{\|T(\gamma)(\epsilon w)\|} \frac{\|T(\gamma)(\epsilon w)\|}{\|T(\gamma)u\|} \right\| \\
= & \left\| \epsilon \frac{T(\gamma)w}{\|T(\gamma)(\epsilon w)\|} - \left(1 - \epsilon - \frac{\|T(\gamma)(\epsilon w)\|}{\|T(\gamma)u\|}\right) \frac{T(\gamma)u}{\|T(\gamma)(\epsilon w)\|} \right\|
\end{aligned}
$$

for all $\gamma \in K$ with $\|T(\gamma)u\| \neq 0$ and sufficiently small ϵ. The last line of the equality above approaches zero as $\epsilon \to 0$. Since $W(\epsilon w) = \epsilon(w - X(w)) = \epsilon W(w)$, as in the case that $O \neq 0$ and $\|W(u)\| = 0$, we have

$$
\begin{aligned}
C & \leq \lim_{\epsilon \searrow 0} \sum_{\gamma \in K} \|T(\gamma)(\epsilon w)\|^{p-1} \frac{j(T(\gamma)(\epsilon w))(T_0(\gamma)W(\epsilon w))}{\|W(\epsilon w)\|} m(\gamma) \\
& = 2 \sum_{\gamma \in K} \|T(\gamma)u\|^{p-1} j(T(\gamma)u) \frac{W(w)}{\|W(w)\|} m(\gamma) \\
& \leq |\nabla_- F_{\alpha',p}|(u).
\end{aligned}
$$

This completes the proof. $\square$

Note that the assumption that Γ is Abelian is used only for the case that u satisfies $\|W(u)\| \neq 0$. Hence, if there exists $C > 0$ satisfying $|\nabla_- F_{\alpha',p}|(u) \geq C$ for all $u \in B$ with $W(u) \neq 0$, we can show Proposition 5 without the assumption.

Chapter 5

A generalization of a result of Żuk

A. Żuk gave a criterion for a finitely generated group to have Property (T) in [Żuk03]. We will generalize the result to the case of uniformly convex and uniformly smooth real Banach spaces.

5.1 Preliminaries

Let Γ be a finitely generated group, K a symmetric finite generating subset of Γ not containing the identity element e of Γ. Let $(B, \| \ \|)$ be a uniformly convex and uniformly smooth real Banach space. Fix p with $1 < p < \infty$ in this chapter.

We set $L(K)$ to be a finite oriented graph whose vertex set is K and edge set is $T := \{(\gamma, \gamma') \in K \times K : \gamma^{-1}\gamma' \ \in K\}$. Note that $(\gamma, \gamma', \gamma^{-1}\gamma') \in K \times K \times K$ if and only if $(\gamma^{-1}, \gamma^{-1}\gamma', \gamma') \in K \times K \times K$. Hence $(\gamma, \gamma') \in T$ if and only if $(\gamma^{-1}, \gamma^{-1}\gamma') \in T$. Set $K^2 = \{\gamma\gamma' \in \Gamma : \gamma, \gamma' \in K\}$. Then $K' := K \cup K^2 \backslash \{e\}$ is a symmetric finite generating subset of Γ, and $L(K')$ is connected. Indeed, all elements in K are connected in $L(K')$ and, for any $\gamma, \gamma' \in K$, $\gamma^{-1}\gamma'$ and γ are connected in $L(K')$. Hence we may assume that $L(K)$ is connected. We denote by $\deg(\gamma)$ the degree of vertex $\gamma \in L(K)$, that is, the number of edges adjacent to γ. Since $L(K)$ is connected, $\deg(\gamma) > 0$ for all $\gamma \in L(K)$. Note that $\deg(\gamma) = \deg(\gamma^{-1})$ for all $\gamma \in L(K)$ and $\sum_{\gamma \in K} \deg(\gamma) = |T|$.

We define ${}^* : B \to B^*$ by $u^* = \|u\|^{p-1} j(u)$ for each $u \in B$. Although * is not linear, $(-v)^* = -v^*$ for all $v \in B$. Besides, by the uniform continuity and the uniform smoothness, * is one-to-one from B onto B^*. Thus we can write $w \in B^*$ as $w = v^*$ by an appropriate $v \in B$. Conversely, any $u \in B$ is written as $(u^*)^*$. For $u \in B$, we say u^* to be the *dual of* u, and u the *dual of* u^*. The map * is continuous. Indeed, the continuity at 0 is obvious. By Proposition 1.3.5, for

$v \in B\backslash\{0\}$, we have

$$\begin{aligned}
& \|v^* - u^*\|_{B^*} \\
= \ & \|\|v\|^{p-1}j(v) - \|u\|^{p-1}j(u)\|_{B^*} \\
\leq \ & \|\|v\|^{p-1}(j(v) - j(u))\|_{B^*} + \|(\|v\|^{p-1} - \|u\|^{p-1})j(u)\|_{B^*} \\
\leq \ & \|v\|^{p-1}\rho_B\left(2\left\|\frac{v}{\|v\|} - \frac{u}{\|u\|}\right\|\right)\Big/\left\|\frac{v}{\|v\|} - \frac{u}{\|u\|}\right\| + |\|v\|^{p-1} - \|u\|^{p-1}|
\end{aligned}$$

for all $u \in B\backslash\{0\}$. Since $\rho_B(\tau)/\tau \to 0$ as $\tau \to 0$, * is continuous at v.

For $u \in B$, $\|u^*\|_{B^*} = \|u\|^{p-1} = \|u\|^{p/q}$, where q is the conjugate exponent of p, that is, $q = p/(p-1)$, which satisfies $1/p+1/q = 1$. Hence $u^*(u) = \|u\|^p = \|u^*\|^q_{B^*}$ for all $u \in B$. Conversely, for $u, v \in B$, if $v^*(u) = \|u\|^p = \|v^*\|^q_{B^*}$, then we have $v^* = u^*$. Indeed, $u = 0$ if and only if $v^* = 0$. If $v^* \neq 0$, since

$$j(v)(u) = v^*(u)/\|v^*\|_{B^*} = \|v^*\|^{q-1}_{B^*} = \|u\|^{(q-1)p/q} = \|u\|^{(1-1/q)p} = \|u\|,$$

we have $j(u) = j(v)$. Hence we obtain

$$v^* = \|v\|^{p-1}j(v) = \|v^*\|^{(p-1)q/p}_{B^*}j(v) = \|u\|^{p-1}j(u) = u^*.$$

We denote $v^*(u)$ by $\langle u, v^*\rangle$ for each $v, u \in B$. Note that $|\langle u, v^*\rangle| \leq \|u\|\|v^*\|_{B^*}$ for all $u, v \in B$.

Let π be a linear isometric representation of Γ on B. For each $\gamma \in \Gamma$, define $\pi^*(\gamma)$ to be an operator satisfying $\langle u, \pi^*(\gamma)v^*\rangle := \langle \pi(\gamma^{-1})u, v^*\rangle$ for $u, v \in B$. Then π^* is a linear isometric representation of Γ on B^*. Indeed, for any $\gamma \in \Gamma$, since $\|\pi(\gamma)v\| = \|v\|$, we have

$$\|v\|\|\pi^*(\gamma)v^*\|_{B^*} \geq \langle \pi(\gamma)v, \pi^*(\gamma)v^*\rangle = \langle v, v^*\rangle = \|v\|^p,$$

that is, $\|\pi^*(\gamma)v^*\|_{B^*} \geq \|v\|^{p-1} = \|v^*\|_{B^*}$ and

$$\|\pi^*(\gamma)v^*\|_{B^*} = \sup_{\|u\|=1}\langle u, \pi^*(\gamma)v^*\rangle = \sup_{\|u\|=1}\langle \pi(\gamma^{-1})u, v^*\rangle \leq \sup_{\|u\|=1}\|u\|\|v^*\|_{B^*} = \|v^*\|_{B^*}$$

for all $v \in B$. Hence $\pi^*(\gamma)$ is an isometry. Since $\pi^*(\gamma)\pi^*(\gamma^{-1})$ is the identity map, $\pi^*(\gamma)$ is surjective. Therefore π^* is a homomorphism from Γ into $O(B^*)$, that is, a linear isometric representation of Γ on B^*. Furthermore, since $\langle \pi(\gamma)v, \pi^*(\gamma)v^*\rangle = \|v\|^p = \|\pi(\gamma)v\|^p$ and $\|\pi^*(\gamma)v^*\|^q_{B^*} = \|v^*\|^q = \|v\|^p$, we have $(\pi(\gamma)v)^* = \pi^*(\gamma)v^*$ for all $v \in B$ and $\gamma \in \Gamma$.

Let M be the linear space of all maps from K to B, and M_* be the linear space of all maps from K to B^*. For $f \in M$, we denote by f^* the map sending $\gamma \in K$ to $(f(\gamma))^* \in B^*$. Since * is one-to-one, every map in M_* is written as f^*

by a certain $f \in M$. Also, every $g \in M$ is written as $(g^*)^*$ by a certain $g^* \in M_*$. We say f^* to be the *dual map of* f, and f the *dual map of* f^*.

We define norms on M and M_* by

$$\|f\|_M := \left(\sum_{\gamma \in K} \|f(\gamma)\|^p \frac{\deg(\gamma)}{|T|} \right)^{1/p} \text{ and } \|g^*\|_{M_*} := \left(\sum_{\gamma \in K} \|g^*(\gamma)\|_{B^*}^q \frac{\deg(\gamma)}{|T|} \right)^{1/q}$$

for each $f \in M$ and $g^* \in M_*$ respectively. Then M and M_* become uniformly convex real Banach spaces (see [Bea85]). We define a bilinear mapping on $M \times M_*$ by

$$\langle f, g^* \rangle := \sum_{\gamma \in K} \langle f(\gamma), g^*(\gamma) \rangle \frac{\deg(\gamma)}{|T|}$$

for each $f \in M$ and $g^* \in M_*$. Using Hölder's inequality, we can easily show that $|\langle f, g^* \rangle| \leq \|f\|_M \|g^*\|_{M_*}$ and $\langle f, f^* \rangle = \|f\|_M^p = \|f^*\|_{M_*}^q$ for $f \in M$ and $g^* \in M_*$.

Lemma 5.1.1. *We can regard $g^* \in M_*$ as a continuous linear functional on M by $g^*(f) := \langle f, g^* \rangle$ for each $f \in M$. This correspondence induces an isometric isomorphism from M_* to M^*, where M^* is the dual Banach space of M.*

Proof. Let $g^* \in M_*$. Obviously, $g^*(\cdot)$ is linear. If $\|g^*\|_{M_*} = 0$, then $\|g^*(\gamma)\|_{B^*} = 0$ for all $\gamma \in K$. Hence

$$\|g^*\|_{M^*} = \sup_{\|f\|_M = 1} |g^*(f)| = \sup_{\|f\|_M = 1} |\langle f, g^* \rangle| \leq \sup_{\|f\|_M = 1} \left| \sum_{\gamma \in K} \langle f(\gamma), g^*(\gamma) \rangle \frac{\deg(\gamma)}{|T|} \right| = 0.$$

Suppose $\|g^*\|_{M_*} \neq 0$. Then $\|g\|_M \neq 0$. We have

$$\begin{aligned} \|g^*\|_{M^*} &= \sup_{\|f\|_M = 1} |\langle f, g^* \rangle| \\ &= \sup_{\|f\|_M = 1} \left| \sum_{\gamma \in K} \langle f(\gamma), g^*(\gamma) \rangle \frac{\deg(\gamma)}{|T|} \right| \\ &\geq \left| \sum_{\gamma \in K} \frac{\langle g(\gamma), g^*(\gamma) \rangle}{\|g\|_M} \frac{\deg(\gamma)}{|T|} \right| \\ &= \frac{\|g^*\|_{M_*}^q}{\|g\|_M}. \end{aligned}$$

Since $\|g\|_M = \|g^*\|_{M_*}^{q/p} = \|g^*\|_{M_*}^{q(1-1/q)} = \|g^*\|_{M_*}^{q-1}$, we obtain $\|g^*\|_{M_*} \leq \|g^*\|_{M^*}$. On the other hand, using Hölder's inequality, we have

$$\begin{aligned}
\|g^*\|_{M^*} &= \sup_{\|f\|_M=1} \left| \sum_{\gamma\in K} \langle f(\gamma), g^*(\gamma)\rangle \frac{\deg(\gamma)}{|T|} \right| \\
&\le \sup_{\|f\|_M=1} \left(\sum_{\gamma\in K} \|f(\gamma)\| \|g^*(\gamma)\|_{B^*} \frac{\deg(\gamma)}{|T|} \right) \\
&\le \sup_{\|f\|_M=1} \left(\sum_{\gamma\in K} \|f(\gamma)\|^p \frac{\deg(\gamma)}{|T|} \right)^{1/p} \left(\sum_{\gamma\in K} \|g^*(\gamma)\|_{B^*}^q \frac{\deg(\gamma)}{|T|} \right)^{1/q} \\
&= \|g^*\|_{M_*}.
\end{aligned}$$

Therefore, $\|g^*\|_{M_*} = \|g^*\|_{M^*}$ for all $g^* \in M_*$, and $g^*(\cdot) \in M^*$. In particular, the correspondence is injective.

Next we prove that any $\tilde{g} \in M^*$ has a map in M_* corresponding to $\tilde{g}$, that is, the correspondence is surjective. For each $v \in B$ and $\gamma \in K$, we define $v_\gamma \in M$ by $v_\gamma(\gamma) = v$ and $v_\gamma(\gamma') = 0$ for other $\gamma' \in K$. Then every $f \in M$ is written as $\sum_{\gamma\in K}(f(\gamma))_\gamma$. We define $g_0^* \in M_*$ by $g_0^*(\gamma)(v) := \tilde{g}(v_\gamma)|T|/\deg(\gamma)$ for each $v \in B$ and $\gamma \in K$. Then, since $\tilde{g}$ is linear, for any $f \in M$, we have

$$\tilde{g}(f) = \sum_{\gamma\in K} \tilde{g}((f(\gamma))_\gamma) = \sum_{\gamma\in K} g_0^*(\gamma)(f(\gamma)) \frac{\deg(\gamma)}{|T|} = \langle f, g_0^* \rangle = g_0^*(f).$$

Therefore g_0^* corresponds to $\tilde{g}$. This completes the proof. $\square$

This lemma implies that M is uniformly smooth, because M_* is uniformly convex and the dual Banach space of a uniformly convex real Banach space is uniformly smooth. Similarly, we can show the following

Lemma 5.1.2. *We can regard $f \in M$ as a continuous linear functional on M_* by $g^*(f) := \langle f, g^* \rangle$ for each $g^* \in M_*$. This correspondence induces an isometric isomorphism from M to $(M_*)^*$.*

Hence M_* is also uniformly smooth. Set

$$\begin{aligned}
C^1 &:= \{f \in M : f(\gamma^{-1}) = -\pi(\gamma^{-1})f(\gamma) \text{ for all } \gamma \in K\}, \\
C^1_* &:= \{g^* \in M_* : g^*(\gamma^{-1}) = -\pi^*(\gamma^{-1})g^*(\gamma) \text{ for all } \gamma \in K\}.
\end{aligned}$$

We can easily see that C^1 is closed in M, and C^1_* is also closed in M_*. Hence C^1 and C^1_* are uniformly convex and uniformly smooth real Banach spaces. Moreover, since every $f \in C^1$ satisfies

$$f^*(\gamma^{-1}) = (f(\gamma^{-1}))^* = (-\pi(\gamma^{-1})f(\gamma))^* = -\pi^*(\gamma^{-1})f^*(\gamma)$$

for all $\gamma \in K$, $f^* \in C^1_*$. Conversely, every map in C^1_* is the dual of a map in C^1.

Lemma 5.1.3. *The correspondence in Lemma 5.1.1 induces an isometric isomorphism from C^1_* onto $(C^1)^*$.*

Proof. For $g^* \in C^1_*$, since $g \in C^1$, as in the proof of Lemma 5.1.1, we have $\|g^*\|_{C^1_*} (= \|g^*\|_{M_*}) = \|g^*\|_{(C^1)^*}$. Hence we should show that any $\tilde{g} \in (C^1)^*$ has a map in C^1_* corresponding to $\tilde{g}$.

We denote by $\operatorname{Hom} B$ the dual linear space of B. Note that we are using the symbol B^* for the dual Banach space, and $\operatorname{Hom} B$ is different from B^* in general. For each $\gamma \in \Gamma$, we set $(\pi^*(\gamma)w)(v) := w(\pi(\gamma^{-1})v)$ for each $w \in \operatorname{Hom} B$ and $v \in B$. Then $\pi^*(\gamma)$ is a linear operator for all $\gamma \in \Gamma$, and π^* is a homomorphism from Γ into the group of all bijective linear operators on $\operatorname{Hom} B$. We can regard C^1 as

$$\left\{ (f(\gamma)) \in \bigoplus_{\gamma \in K} B_\gamma \ : \ f(\gamma^{-1}) = -\pi(\gamma^{-1}) f(\gamma) \text{ for all } \gamma \in K \right\},$$

where B_γ is a copy of B indexed by each $\gamma \in K$. Since K is a finite set, we have $\operatorname{Hom}(\bigoplus_{\gamma \in K} B_\gamma) = \bigoplus_{\gamma \in K} \operatorname{Hom} B_\gamma$. Hence $\operatorname{Hom} C^1$ is isomorphic to

$$\bigoplus_{\gamma \in K} \operatorname{Hom} B_\gamma \Big/ \left\{ (h(\gamma)) \in \bigoplus_{\gamma \in K} \operatorname{Hom} B_\gamma : \sum_{\gamma \in K} h(\gamma)(f(\gamma)) = 0 \text{ for all } f \in C^1 \right\}.$$

For an arbitrary $\gamma_0 \in K$, we take $f \in C^1$ such that $\|f(\gamma)\| = 0$ unless $\gamma = \gamma_0$ or $\gamma = \gamma_0^{-1}$. If $\gamma_0 \neq \gamma_0^{-1}$, then we have

$$\begin{aligned}
\sum_{\gamma \in K} h(\gamma)(f(\gamma)) &= h(\gamma_0)(f(\gamma_0)) + h(\gamma_0^{-1})(f(\gamma_0^{-1})) \\
&= h(\gamma_0)(f(\gamma_0)) + h(\gamma_0^{-1})(-\pi(\gamma_0^{-1}) f(\gamma_0)) \\
&= h(\gamma_0)(f(\gamma_0)) - (\pi^*(\gamma_0) h(\gamma_0^{-1}))(f(\gamma_0)) \\
&= (h(\gamma_0) - \pi^*(\gamma_0) h(\gamma_0^{-1}))(f(\gamma_0))
\end{aligned}$$

for all $(h(\gamma)) \in \bigoplus_{\gamma \in K} \operatorname{Hom} B_\gamma$. If $\gamma_0 = \gamma_0^{-1}$, then we have

$$\begin{aligned}
\sum_{\gamma \in K} h(\gamma)(f(\gamma)) &= \frac{1}{2} \left(h(\gamma_0)(f(\gamma_0)) + h(\gamma_0^{-1})(f(\gamma_0^{-1})) \right) \\
&= \frac{1}{2} \left((h(\gamma_0) - \pi^*(\gamma_0) h(\gamma_0^{-1}))(f(\gamma_0)) \right)
\end{aligned}$$

for all $(h(\gamma)) \in \bigoplus_{\gamma \in K} \operatorname{Hom} B_\gamma$. Any map in C^1 is described as the sum of maps such that the values of each of these are trivial except for a certain element in K and its inverse. Hence $(h(\gamma)) \in \bigoplus_{\gamma \in K} \operatorname{Hom} B_\gamma$ satisfies $\sum_{\gamma \in K} h(\gamma)(f(\gamma)) = 0$ for all $f \in C^1$ if and only if it satisfies $h(\gamma) = \pi^*(\gamma)h(\gamma^{-1})$ for all $\gamma \in K$. Therefore $\operatorname{Hom} C^1$ is isomorphic to

$$\bigoplus_{\gamma \in K} \operatorname{Hom} B_\gamma \Big/ \left\{ (h(\gamma)) : h(\gamma) = \pi^*(\gamma)h(\gamma^{-1}) \text{ for all } \gamma \in K \right\}.$$

Every $(h(\gamma)) \in \bigoplus_{\gamma \in K} \operatorname{Hom} B_\gamma$ is decomposed as

$$(h(\gamma)) = \left(\frac{h(\gamma) - \pi^*(\gamma)h(\gamma^{-1})}{2} \right) + \left(\frac{h(\gamma) + \pi^*(\gamma)h(\gamma^{-1})}{2} \right),$$

and satisfies

$$\pi^*(\gamma^{-1}) \left(\frac{h(\gamma) - \pi^*(\gamma)h(\gamma^{-1})}{2} \right) = -\frac{h(\gamma^{-1}) - \pi^*(\gamma^{-1})h(\gamma)}{2}$$

and

$$\pi^*(\gamma^{-1}) \left(\frac{h(\gamma) + \pi^*(\gamma)h(\gamma^{-1})}{2} \right) = \frac{h(\gamma^{-1}) + \pi^*(\gamma^{-1})h(\gamma)}{2}.$$

On the other hand, for $(h(\gamma)) \in \bigoplus_{\gamma \in K} \operatorname{Hom} B_\gamma$, $h(\gamma) = -\pi^*(\gamma)h(\gamma^{-1})$ and $h(\gamma) = \pi^*(\gamma)h(\gamma^{-1})$ for all $\gamma \in K$ if and only if $(h(\gamma)) = 0$ for all $\gamma \in K$. Therefore $\bigoplus_{\gamma \in K} \operatorname{Hom} B_\gamma$ is isomorphic to

$$\left\{ (h(\gamma)) : h(\gamma) = \pi^*(\gamma)h(\gamma^{-1}) \right\} \oplus \left\{ (h(\gamma)) : h(\gamma) = -\pi^*(\gamma)h(\gamma^{-1}) \right\},$$

and hence $\operatorname{Hom} C^1$ is isomorphic to

$$\left\{ (h(\gamma)) \in \bigoplus_{\gamma \in K} \operatorname{Hom} B_\gamma : h(\gamma) = -\pi^*(\gamma)h(\gamma^{-1}) \text{ for all } \gamma \in K \right\}.$$

Let $\tilde{g} \in (C^1)^*$. For each $v \in B$ and $\gamma \in K$, we define $v_\gamma \in M$ by $v_\gamma(\gamma) = v$ and $v_\gamma(\gamma') = 0$ for other $\gamma' \in K$. We define $g_0^* \in M_*$ by $g_0^*(\gamma)(v) := \tilde{g}(v_\gamma)|T|/\deg(\gamma)$ for each $v \in B$ and $\gamma \in K$. Then, as the proof of Lemma 5.1.1, we can show that $\tilde{g}(f) = g_0^*(f)$ for all $f \in C^1$. Because $\tilde{g} \in \operatorname{Hom} C^1$, $g_0^* \in \operatorname{Hom} C^1$. Regarding $\operatorname{Hom} C^1$ as a subspace of $\bigoplus_{\gamma \in K} \operatorname{Hom} B_\gamma$ as above, we see that $g_0^* \in \operatorname{Hom} C^1 \cap M_* = C_*^1$. Therefore $\tilde{g} = g_0^*$. This completes the proof. □

Let C^2 be the linear space of all maps from T to B, and C_*^2 the linear space of all maps from T to B^*. As in the case of M_*, for $h \in C^2$, we denote by h^* the dual map in C_*^2 sending $(\gamma, \gamma') \in T$ to $(h(\gamma, \gamma'))^* \in B^*$. We define norms on C^2 and C_*^2 by

$$\|h\|_{C^2} := \left(\sum_{t \in T} \frac{\|f(t)\|^p}{|T|} \right)^{1/p} \text{ and } \|l^*\|_{C_*^2} := \left(\sum_{t \in T} \frac{\|l^*(t)\|_{B^*}^q}{|T|} \right)^{1/q},$$

then C^2 and C^2_* become uniformly convex and uniformly smooth real Banach spaces as M and M_*. We define a bilinear mapping on $C^2 \times C^2_*$ by

$$\langle h, l^* \rangle := \sum_{t \in T} \frac{\langle h(t), l^*(t) \rangle}{|T|}$$

for each $h \in C^2$ and $l^* \in C^2_*$.

We define a linear operator $d^0 : B \to C^1$ by $(d^0 u)(\gamma) := \pi(\gamma)u - u$ for each $u \in B$ and $\gamma \in K$. We also define a linear operator $d^1 : C^1 \to C^2$ by $(d^1 f)(\gamma, \gamma') := f(\gamma) - f(\gamma') + \pi(\gamma) f(\gamma^{-1}\gamma')$ for each $f \in C^1$ and $(\gamma, \gamma') \in T$. Similarly, we define a linear operator $\delta^0 : B^* \to C^1_*$ by $(\delta^0 v^*)(\gamma) := \pi^*(\gamma)v^* - v^*$ for each $v^* \in B^*$ and $\gamma \in K$. We also define a linear operator $\delta^1 : C^1_* \to C^2_*$ by $(\delta^1 g^*)(\gamma, \gamma') := g^*(\gamma) - g^*(\gamma') + \pi^*(\gamma) g^*(\gamma^{-1}\gamma')$ for each $g^* \in C^1_*$ and $(\gamma, \gamma') \in T$.

Lemma 5.1.4. *$d^1 \circ d^0 = 0$.*

Proof. Every $u \in B$ satisfies

$$\begin{aligned} (d^1 \circ d^0 u)(\gamma, \gamma') &= d^1(\pi(\cdot)u - u)(\gamma, \gamma') \\ &= (\pi(\gamma)u - u) - (\pi(\gamma')u - u) + \pi(\gamma)(\pi(\gamma^{-1}\gamma')u - u) \\ &= 0 \end{aligned}$$

for all $(\gamma, \gamma') \in T$, that is, $d^1 \circ d^0 = 0$. $\square$

Lemma 5.1.5. *$\|d^1\| \le 3$.*

Proof. For $f \in C^1$ we have

$$\begin{aligned} \sum_{(\gamma,\gamma') \in T} \frac{\|f(\gamma^{-1}\gamma')\|^p}{|T|} &= \sum_{(\gamma^{-1},\gamma^{-1}\gamma') \in T} \frac{\|f(\gamma^{-1}\gamma')\|^p}{|T|} \\ &= \sum_{(\gamma''(\gamma')^{-1},\gamma'') \in T} \frac{\|f(\gamma'')\|^p}{|T|} \\ &= \sum_{\gamma'' \in K} \|f(\gamma'')\|^p \frac{\deg(\gamma'')}{|T|}, \end{aligned}$$

where we write $\gamma^{-1}\gamma'$ as γ''. Therefore we have

$$\begin{aligned}
& \|d^1 f\|_{C^2} \\
= & \left(\sum_{(\gamma,\gamma')\in T} \frac{\|f(\gamma) - f(\gamma') + \pi(\gamma) f(\gamma^{-1}\gamma')\|^p}{|T|}\right)^{1/p} \\
\le & \left(\sum_{(\gamma,\gamma')\in T} \frac{(\|f(\gamma)\| + \|f(\gamma')\| + \|\pi(\gamma) f(\gamma^{-1}\gamma')\|)^p}{|T|}\right)^{1/p} \\
\le & \left(\sum_{(\gamma,\gamma')\in T} \frac{\|f(\gamma)\|^p}{|T|}\right)^{\frac{1}{p}} + \left(\sum_{(\gamma,\gamma')\in T} \frac{\|f(\gamma')\|^p}{|T|}\right)^{\frac{1}{p}} + \left(\sum_{(\gamma,\gamma')\in T} \frac{\|f(\gamma^{-1}\gamma')\|^p}{|T|}\right)^{\frac{1}{p}} \\
= & \ 3\left(\sum_{\gamma\in K} \|f(\gamma)\|^p \frac{\deg(\gamma)}{|T|}\right)^{\frac{1}{p}} \\
= & \ 3\|f\|_M.
\end{aligned}$$

$\square$

Let d^* be the adjoint operator of d^0 defined by $\langle u, d^* g^*\rangle := \langle d^0 u, g^*\rangle$ for each $u \in B$ and $g^* \in C^1_*$. Let δ^* be the adjoint operator of δ^0 defined by $\langle \delta^* f, v^*\rangle := \langle f, \delta^0 v^*\rangle$ for each $v^* \in B^*$ and $f \in C^1$.

Lemma 5.1.6. *For $g^* \in C^1_*$ and $f \in C^1$*

$$d^* g^* = -2\sum_{\gamma\in K} g^*(\gamma)\frac{\deg(\gamma)}{|T|} \quad \text{and} \quad \delta^* f = -2\sum_{\gamma\in K} f(\gamma)\frac{\deg(\gamma)}{|T|}.$$

Proof. For $u \in B$ and $\gamma \in K$

$$\begin{aligned}
\langle d^0 u(\gamma), g^*(\gamma)\rangle &= \langle \pi(\gamma)u - u, g^*(\gamma)\rangle \\
&= \langle u, \pi^*(\gamma^{-1}) g^*(\gamma)\rangle - \langle u, g^*(\gamma)\rangle \\
&= \langle u, -g^*(\gamma^{-1})\rangle - \langle u, g^*(\gamma)\rangle.
\end{aligned}$$

Since K is symmetric and $\deg(\gamma) = \deg(\gamma^{-1})$ for all $\gamma \in K$, we have

$$\begin{aligned}
\langle d^0 u, g^*\rangle &= \sum_{\gamma\in K}\langle d^0 u(\gamma), g^*(\gamma)\rangle \frac{\deg(\gamma)}{|T|} \\
&= \sum_{\gamma\in K}\langle u, -g^*(\gamma^{-1})\rangle \frac{\deg(\gamma)}{|T|} - \sum_{\gamma\in K}\langle u, g^*(\gamma)\rangle \frac{\deg(\gamma)}{|T|} \\
&= -2\sum_{\gamma\in K}\langle u, g^*(\gamma)\rangle \frac{\deg(\gamma)}{|T|}
\end{aligned}$$

for all $u \in B$. This implies $\langle u, d^* g^* \rangle = \langle u, -2\sum_{\gamma\in K} g^*(\gamma)\deg(\gamma)/|T|\rangle$ for all $u \in B$, hence the expression of d^* is obtained. The proof of the expression of δ^* is the same as that of the expression of d^*. $\square$

Lemma 5.1.7. $\|\delta^*\| \leq 2$.

Proof. For $f \in C^1$, we have

$$\begin{aligned}
\|\delta^* f\| &= \left\| -2\sum_{\gamma\in K} f(\gamma)\frac{\deg(\gamma)}{|T|} \right\| \\
&\leq 2\left(\sum_{\gamma\in K} \|f(\gamma)\| \frac{\deg(\gamma)}{|T|} \right) \\
&\leq 2\left(\sum_{\gamma\in K} \|f(\gamma)\|^p \frac{\deg(\gamma)}{|T|} \right)^{1/p} \left(\sum_{\gamma\in K} \frac{\deg(\gamma)}{|T|} \right)^{1/q} \\
&= 2\|f\|_M.
\end{aligned}$$

Hence $\|\delta^*\| \leq 2$. $\square$

Let B^1 be the kernel of d^1.

Proposition 5.1.8. *Suppose that π has no non-trivial invariant vector. If there exists $C > 0$ such that $\langle \delta^* f, d^* f^* \rangle \geq C\langle f, f^* \rangle$ for all $f \in B^1$, then we have*

$$\max_{\gamma\in K} \|(d^0 u)(\gamma)\| = \max_{\gamma\in K} \|\pi(\gamma)u - u\| \geq \frac{C}{2}\|u\|$$

for all $u \in B$.

Proof. If B is 0-dimentional, then the proposition is obvious. Suppose that B is not 0-dimentional. Since π has no non-trivial invariant vector, $d^0(B)$ contains a non-trivial vector. Since $d^0(B) \subset B^1$ by Lemma 5.1.4, B^1 also contains a non-trivial vector.

First, we prove that the assumption implies that $d^0\delta^* : B^1 \to B^1$ has a bounded inverse. By Lemma 5.1.5, d^1 is bounded, hence B^1 is a closed subspace of C^1. By the assumption of the proposition, we have

$$C\|f\|_M^p = C\langle f, f^* \rangle \leq \langle d^0\delta^* f, f^* \rangle \leq \|d^0\delta^* f\|_M \|f^*\|_{M_*}$$

for all $f \in B^1$. Since $\|f^*\|_{M_*} = \|f\|_M^{p/q} = \|f\|_M^{p(1-1/p)} = \|f\|_M^{p-1}$, we have $C\|f\|_M \leq \|d^0\delta^* f\|_M$ for all $f \in B^1$. This implies that $d^0\delta^*(B^1)$ is closed in B^1.

Suppose that $d^0\delta^*(B^1)$ is properly contained in B^1. Then there exists $f_0 \in B^1$ such that $\|f_0 - d^0\delta^* B^1\| > 0$. Hence, by Hahn-Banach theorem, there exists $\tilde{g}_1 \in (B^1)^*$ such that

$$\|\tilde{g}_1\|_{(B^1)^*} = 1, \ \tilde{g}_1(f_0) = \|f_0 - d^0\delta^* B^1\| \ \text{ and } \ \tilde{g}_1(d^0\delta^* f) = 0$$

for all $f \in B^1$. Thus, again by Hahn-Banach theorem, there exists $g_2^* \in (C^1)^*$ such that

$$\|g_2^*\|_{(C^1)^*} = \|\tilde{g}_1\|_{(B^1)^*} \ \text{ and } \ g_2^*(f) = \tilde{g}_1(f)$$

for all $f \in B^1$. In particular, we have $g_2^*(d^0\delta^* f) = 0$ for all $f \in B^1$. On the other hand, since B^1 is a uniformly convex and uniformly smooth real Banach space, we can take $g_1 \in B^1$ satisfying $\tilde{g}_1(g_1) = \|\tilde{g}_1\|^q_{(B^1)^*} = \|g_1\|^p_M$. Hence we have $g_2^*(g_1) = \tilde{g}_1(g_1) = \|g_1\|^p_M$. Since $(C^1)^*$ is isometrically isomorphic to C^1_*, which is a subspace of M_*, $\|g_2^*\|_{(C^1)^*} = \|g_2^*\|_{C^1_*} = \|g_2^*\|_{M_*}$. Therefore $g_2^*(g_1) = \|g_1\|^p_M = \|g_2^*\|^q_{M_*}$. Since the dual map of g_2^* is unique by the smoothness of M_*, g_1 must be coincide with g_2. By the assumption of the proposition, we have

$$0 = g_2^*(d^0\delta^* g_1) = \langle d^0\delta^* g_1, g_2^*\rangle \geq C\langle g_1, g_2^*\rangle = C\|g_1\|^p_{C^1} = C\|\tilde{g}_1\|^q_{(B^1)^*} = C > 0.$$

This is a contradiction. Thus $d^0\delta^*(B^1) = B^1$, that is, $d^0\delta^*$ is surjective.

Since $C\|f\|_M \leq \|d^0\delta^* f\|_M$ for all $f \in B^1$, $d^0\delta^*$ is bijective and has the inverse $(d^0\delta^*)^{-1} : B^1 \to B^1$. Moreover, we have $\|(d^0\delta^*)^{-1}\|_{B^1 \to B^1} \leq C^{-1}$, where $\| \ \|_{B_1 \to B_2}$ denotes the operator norm of an operator from a Banach space B_1 into a Banach space B_2. Hence $(d^0\delta^*)^{-1}$ is bounded.

We prove the proposition by contradiction. Suppose that there exists a non-trivial $u \in B$ satisfying

$$\max_{\gamma \in K} \|(d^0 u)(\gamma)\| < \frac{C}{2}\|u\|.$$

Then we have

$$\|d^0 u\|^p_M = \sum_{\gamma \in K} \|(d^0 u)(\gamma)\|^p \frac{\deg(\gamma)}{|T|} < \sum_{\gamma \in K} \left(\frac{C}{2}\|u\|\right)^p \frac{\deg(\gamma)}{|T|} = \left(\frac{C}{2}\|u\|\right)^p,$$

which gives $\|d^0 u\|_M < C\|u\|/2$. Let us consider $\delta^*(d^0\delta^*)^{-1} d^0 u \in B$. We obtain

$$\|\delta^*(d^0\delta^*)^{-1} d^0 u\| \leq \|\delta^*\|_{B^1 \to B}\|(d^0\delta^*)^{-1}\|_{B^1 \to B^1}\|d^0 u\|_{B^1} < 2C^{-1}C\|u\|/2 = \|u\|,$$

where $\| \ \|_{B^1} = \| \ \|_M$. Thus $\delta^*(d^0\delta^*)^{-1} d^0 u \neq u$. On the other hand, we have

$$d^0(\delta^*(d^0\delta^*)^{-1} d^0 u - u) = d^0\delta^*(d^0\delta^*)^{-1} d^0 u - d^0 u = 0.$$

Hence $u' := \delta^*(d^0\delta^*)^{-1} d^0 u - u \neq 0$ satisfies $\pi(\gamma)u' - u' = 0$ for all $\gamma \in K$, that is, u' is a non-trivial invariant vector. This contradicts the assumption on π. This proves the proposition. $\square$

5.2 Proof of Theorem 6

Let us define $D : C^1 \to C^2$ and $D^* : C^1_* \to C^2_*$ as follows:

$$(Df)(\gamma, \gamma') := f(\gamma) - f(\gamma'),\ (D^* f^*)(\gamma, \gamma') := f^*(\gamma) - f^*(\gamma')$$

for each $f \in C^1$, $f^* \in C^1_*$ and $(\gamma, \gamma') \in T$.

Proposition 5.2.1. *Every $f \in C^1$ satisfies*

$$\langle Df, D^* f^* \rangle = \langle f, f^* \rangle + \frac{1}{3} \langle d^1 f, \delta^1 f^* \rangle.$$

Proof. We have

$$d^1 f(\gamma, \gamma') = f(\gamma) - f(\gamma') + \pi(\gamma) f(\gamma^{-1}\gamma') = (Df)(\gamma, \gamma') + \pi(\gamma) f(\gamma^{-1}\gamma')$$

and similarly

$$\delta^1 f^*(\gamma, \gamma') = (D^* f^*)(\gamma, \gamma') + \pi^*(\gamma) f^*(\gamma^{-1}\gamma')$$

for all $(\gamma, \gamma') \in T$. These implies

$$\begin{aligned}
& \langle (Df)(\gamma, \gamma'), (D^* f^*)(\gamma, \gamma') \rangle \\
= \ & \langle d^1 f(\gamma, \gamma') - \pi(\gamma) f(\gamma^{-1}\gamma'), \delta^1 f^*(\gamma, \gamma') - \pi^*(\gamma) f^*(\gamma^{-1}\gamma') \rangle \\
= \ & \langle d^1 f(\gamma, \gamma'), \delta^1 f^*(\gamma, \gamma') \rangle - \langle d^1 f(\gamma, \gamma'), \pi^*(\gamma) f^*(\gamma^{-1}\gamma') \rangle \\
& - \langle \pi(\gamma) f(\gamma^{-1}\gamma'), \delta^1 f^*(\gamma, \gamma') \rangle + \langle \pi(\gamma) f(\gamma^{-1}\gamma'), \pi^*(\gamma) f^*(\gamma^{-1}\gamma') \rangle \\
= \ & \langle d^1 f(\gamma, \gamma'), \delta^1 f^*(\gamma, \gamma') \rangle - \langle d^1 f(\gamma, \gamma'), \pi^*(\gamma) f^*(\gamma^{-1}\gamma') \rangle \\
& - \langle \pi(\gamma) f(\gamma^{-1}\gamma'), \delta^1 f^*(\gamma, \gamma') \rangle + \langle f(\gamma^{-1}\gamma'), f^*(\gamma^{-1}\gamma') \rangle
\end{aligned}$$

for all $(\gamma, \gamma') \in T$. Thus we obtain

$$\begin{aligned}
& \langle Df, D^* f^* \rangle \\
= \ & \sum_{(\gamma, \gamma') \in T} \langle (Df)(\gamma, \gamma'), (D^* f^*)(\gamma, \gamma') \rangle \frac{1}{|T|} \\
= \ & \langle d^1 f, \delta^1 f^* \rangle - \sum_{(\gamma, \gamma') \in T} \langle d^1 f(\gamma, \gamma'), \pi^*(\gamma) f^*(\gamma^{-1}\gamma') \rangle \frac{1}{|T|} \\
& - \sum_{(\gamma, \gamma') \in T} \langle \pi(\gamma) f(\gamma^{-1}\gamma'), \delta^1 f^*(\gamma, \gamma') \rangle \frac{1}{|T|} + \sum_{(\gamma, \gamma') \in T} \langle f(\gamma^{-1}\gamma'), f^*(\gamma^{-1}\gamma') \rangle \frac{1}{|T|}.
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{(\gamma,\gamma')\in T}\langle f(\gamma^{-1}\gamma'), f^*(\gamma^{-1}\gamma')\rangle\frac{1}{|T|} &= \sum_{(\gamma^{-1},\gamma^{-1}\gamma')\in T}\langle f(\gamma^{-1}\gamma'), f^*(\gamma^{-1}\gamma')\rangle\frac{1}{|T|} \\
&= \sum_{(\gamma''(\gamma')^{-1},\gamma'')\in T}\langle f(\gamma''), f^*(\gamma'')\rangle\frac{1}{|T|} \\
&= \sum_{\gamma''\in K}\langle f(\gamma''), f^*(\gamma'')\rangle\frac{\deg(\gamma)}{|T|} \\
&= \langle f, f^*\rangle.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
d^1f(\gamma,\gamma') &= f(\gamma) - f(\gamma') + \pi(\gamma)f(\gamma^{-1}\gamma') \\
&= -\pi(\gamma)(-\pi(\gamma^{-1})f(\gamma) + \pi(\gamma^{-1})f(\gamma') - f(\gamma^{-1}\gamma')) \\
&= -\pi(\gamma)(f(\gamma^{-1}) - f(\gamma^{-1}\gamma') + \pi(\gamma^{-1})f(\gamma(\gamma^{-1}\gamma'))) \\
&= -\pi(\gamma)d^1f(\gamma^{-1},\gamma^{-1}\gamma'),
\end{aligned}$$

we get

$$\begin{aligned}
\langle d^1f(\gamma,\gamma'), \pi^*(\gamma)f^*(\gamma^{-1}\gamma')\rangle &= \langle -\pi(\gamma)d^1f(\gamma^{-1},\gamma^{-1}\gamma'), \pi^*(\gamma)f^*(\gamma^{-1}\gamma')\rangle \\
&= -\langle d^1f(\gamma^{-1},\gamma^{-1}\gamma'), f^*(\gamma^{-1}\gamma')\rangle
\end{aligned}$$

for all $(\gamma,\gamma')\in T$. Thus we have

$$\begin{aligned}
\sum_{(\gamma,\gamma')\in T}\langle d^1f(\gamma,\gamma'), \pi^*(\gamma)f^*(\gamma^{-1}\gamma')\rangle &= -\sum_{(\gamma,\gamma')\in T}\langle d^1f(\gamma^{-1},\gamma^{-1}\gamma'), f^*(\gamma^{-1}\gamma')\rangle \\
&= -\sum_{(\gamma^{-1},\gamma^{-1}\gamma')\in T}\langle d^1f(\gamma^{-1},\gamma^{-1}\gamma'), f^*(\gamma^{-1}\gamma')\rangle \\
&= -\sum_{(\gamma,\gamma')\in T}\langle d^1f(\gamma,\gamma'), f^*(\gamma')\rangle.
\end{aligned}$$

Also, since

$$\begin{aligned}
d^1f(\gamma,\gamma') &= f(\gamma) - f(\gamma') + \pi(\gamma)f(\gamma^{-1}\gamma') \\
&= -(f(\gamma') - f(\gamma) - \pi(\gamma)(-\pi(\gamma^{-1}\gamma')f((\gamma')^{-1}\gamma))) \\
&= -(f(\gamma') - f(\gamma) + \pi(\gamma')f((\gamma')^{-1}\gamma)) \\
&= -d^1f(\gamma',\gamma)
\end{aligned}$$

for all $(\gamma,\gamma')\in T$, we obtain

$$\begin{aligned}
\sum_{(\gamma,\gamma')\in T}\langle d^1f(\gamma,\gamma'), \pi^*(\gamma)f^*(\gamma^{-1}\gamma')\rangle &= \sum_{(\gamma,\gamma')\in T}\langle d^1f(\gamma',\gamma), f^*(\gamma')\rangle \\
&= \sum_{(\gamma,\gamma')\in T}\langle d^1f(\gamma,\gamma'), f^*(\gamma)\rangle.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
& \sum_{(\gamma,\gamma')\in T} \langle d^1 f(\gamma,\gamma'), \pi^*(\gamma) f^*(\gamma^{-1}\gamma')\rangle \frac{1}{|T|} \\
= & \frac{1}{3} \sum_{(\gamma,\gamma')\in T} \langle d^1 f(\gamma,\gamma'), \pi^*(\gamma) f^*(\gamma^{-1}\gamma') - f^*(\gamma') + f^*(\gamma)\rangle \frac{1}{|T|} \\
= & \frac{1}{3} \sum_{(\gamma,\gamma')\in T} \langle d^1 f(\gamma,\gamma'), \delta^1 f^*(\gamma,\gamma')\rangle \frac{1}{|T|} \\
= & \frac{1}{3}\langle d^1 f, \delta^1 f^*\rangle.
\end{aligned}$$

Similarly we get

$$\sum_{(\gamma,\gamma')\in T} \langle \pi(\gamma) f(\gamma^{-1}\gamma'), \delta^1 f^*(\gamma,\gamma')\rangle \frac{1}{|T|} = \frac{1}{3}\langle d^1 f, \delta^1 f^*\rangle.$$

Therefore, we obtain

$$\begin{aligned}
\langle Df, D^* f^*\rangle &= \langle d^1 f, \delta^1 f^*\rangle - \frac{1}{3}\langle d^1 f, \delta^1 f^*\rangle - \frac{1}{3}\langle d^1 f, \delta^1 f^*\rangle + \langle f, f^*\rangle \\
&= \langle f, f^*\rangle + \frac{1}{3}\langle d^1 f, \delta^1 f^*\rangle.
\end{aligned}$$

□

Let Δ_B be a discrete Laplacian acting on M defined as follows: For $f \in M$

$$(\Delta_B f)(\gamma) := f(\gamma) - \frac{1}{\deg(\gamma)} \sum_{(\gamma,\gamma')\in T} f(\gamma')$$

for each $\gamma \in K$. We define $P : M \to M$ by

$$P(f)(\gamma) := f(\gamma) + \frac{\delta^* f}{2} = f(\gamma) - \sum_{\gamma'\in K} f(\gamma') \frac{\deg(\gamma')}{|T|}$$

for each $f \in M$ and $\gamma \in K$. Similarly, we define a map $P^* : M_* \to M_*$ by

$$P^*(g^*)(\gamma) := g^*(\gamma) + \frac{d^* g^*}{2} = g^*(\gamma) - \sum_{\gamma'\in K} g^*(\gamma') \frac{\deg(\gamma')}{|T|}$$

for each $g^* \in M_*$ and $\gamma \in K$.

Lemma 5.2.2. *The map P is a projection from M into*

$$\{f \in M : \langle f, g^*\rangle = 0 \text{ for any constant map } g^* \in M_*\}.$$

Similarly, P^* is also a projection from M_* into

$$\{g^* \in M_* : \langle f, g^*\rangle = 0 \text{ for any constant map } f \in M\}.$$

Proof. For $f \in M$, we have

$$\begin{aligned}
& P(P(f))(\gamma) \\
= \;& P(f)(\gamma) - \sum_{\gamma' \in K} P(f)(\gamma') \frac{\deg(\gamma')}{|T|} \\
= \;& f(\gamma) - \sum_{\gamma'' \in K} f(\gamma'') \frac{\deg(\gamma'')}{|T|} - \sum_{\gamma' \in K} \left(f(\gamma') - \sum_{\gamma'' \in K} f(\gamma'') \frac{\deg(\gamma'')}{|T|} \right) \frac{\deg(\gamma')}{|T|} \\
= \;& f(\gamma) - \sum_{\gamma' \in K} f(\gamma') \frac{\deg(\gamma')}{|T|} \\
= \;& P(f)(\gamma)
\end{aligned}$$

for all $\gamma \in K$, that is, P is a projection. On the other hand, let $f \in M$, and let $g^* \in M_*$ be a constant map, that is, a map satisfying $g^*(\gamma) = v^*$ for some vector $v^* \in B^*$ and all $\gamma \in K$. Then we have

$$\begin{aligned}
\langle P(f), g^* \rangle &= \sum_{\gamma \in K} \left\langle f(\gamma) - \sum_{\gamma' \in K} f(\gamma') \frac{\deg(\gamma')}{|T|}, g^*(\gamma) \right\rangle \frac{\deg(\gamma)}{|T|} \\
&= \sum_{\gamma \in K} \langle f(\gamma), v^* \rangle \frac{\deg(\gamma)}{|T|} - \sum_{\gamma \in K} \left\langle \sum_{\gamma' \in K} f(\gamma') \frac{\deg(\gamma')}{|T|}, v^* \right\rangle \frac{\deg(\gamma)}{|T|} \\
&= \sum_{\gamma \in K} \langle f(\gamma), v^* \rangle \frac{\deg(\gamma)}{|T|} - \sum_{\gamma' \in K} \langle f(\gamma'), v^* \rangle \frac{\deg(\gamma')}{|T|} \\
&= 0.
\end{aligned}$$

This proves the lemma for P. A similar computation gives a proof for the case of P^*. □

Let us introduce an invariant of $L(K)$, which is the smallest positive eigenvalue of the discrete Laplacian Δ_B if B is a Hilbert space and $p = 2$.

Definition 5.2.3. We define

$$\lambda_{B,p}(L(K)) := \inf_{f \in M, \langle P(f), P^*(f^*) \rangle \neq 0} \frac{\langle \Delta_B(P(f)), P^*(f^*) \rangle}{\langle P(f), P^*(f^*) \rangle}.$$

We emphasize that $\lambda_{B,p}(L(K))$ is independent of the linear isometric representations of Γ on B.

Note that $\langle \Delta_B(P(f)), P^*(f^*) \rangle \geq 0$ and $\langle P(f), P^*(f^*) \rangle \geq 0$ for all $f \in M$.

Indeed, for $\gamma \in K$,

$$\begin{aligned}
\Delta_B(P(f))(\gamma) &= P(f)(\gamma) - \frac{1}{\deg(\gamma)} \sum_{(\gamma,\gamma')\in T} P(f)(\gamma') \\
&= f(\gamma) + \frac{\delta^* f}{2} - \frac{1}{\deg(\gamma)} \sum_{(\gamma,\gamma')\in T} \left(f(\gamma') + \frac{\delta^* f}{2} \right) \\
&= f(\gamma) - \frac{1}{\deg(\gamma)} \sum_{(\gamma,\gamma')\in T} f(\gamma').
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\langle \Delta_B(P(f)), P^*(f^*) \rangle \\
=\ & \sum_{\gamma\in K} \left\langle f(\gamma) - \frac{1}{\deg(\gamma)} \sum_{(\gamma,\gamma')\in T} f(\gamma'), f^*(\gamma) - \sum_{\gamma''\in K} f^*(\gamma'') \frac{\deg(\gamma'')}{|T|} \right\rangle \frac{\deg(\gamma)}{|T|} \\
=\ & \|f\|_M^p - \sum_{\gamma\in K} \left\langle f(\gamma), \sum_{\gamma''\in K} f^*(\gamma'') \frac{\deg(\gamma'')}{|T|} \right\rangle \frac{\deg(\gamma)}{|T|} \\
& - \sum_{\gamma\in K} \left\langle \sum_{(\gamma,\gamma')\in T} f(\gamma'), f^*(\gamma) \right\rangle \frac{1}{|T|} \\
& + \sum_{\gamma\in K} \left\langle \sum_{(\gamma,\gamma')\in T} f(\gamma'), \sum_{\gamma''\in K} f^*(\gamma'') \frac{\deg(\gamma'')}{|T|} \right\rangle \frac{1}{|T|}.
\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}
& \sum_{\gamma\in K} \left\langle \sum_{(\gamma,\gamma')\in T} f(\gamma'), f^*(\gamma) \right\rangle \frac{1}{|T|} \\
=\ & \sum_{(\gamma,\gamma')\in T} \langle f(\gamma'), f^*(\gamma) \rangle \frac{1}{|T|} \\
\leq\ & \sum_{(\gamma,\gamma')\in T} \|f(\gamma')\| \|f^*(\gamma)\|_{B^*} \frac{1}{|T|} \\
\leq\ & \left(\sum_{(\gamma,\gamma')\in T} \|f(\gamma')\|^p \frac{1}{|T|} \right)^{1/p} \left(\sum_{(\gamma,\gamma')\in T} \|f^*(\gamma)\|_{B^*}^q \frac{1}{|T|} \right)^{1/q} \\
\leq\ & \|f\|_M \|f^*\|_{M_*} \\
=\ & \|f\|_M^p
\end{aligned}$$

and

$$
\begin{aligned}
& \sum_{\gamma\in K}\left\langle \sum_{(\gamma,\gamma')\in T} f(\gamma'), \sum_{\gamma''\in K} f^*(\gamma'')\frac{\deg(\gamma'')}{|T|}\right\rangle \frac{1}{|T|} \\
= & \sum_{\gamma\in K}\left\langle f(\gamma), \sum_{\gamma''\in K} f^*(\gamma'')\frac{\deg(\gamma'')}{|T|}\right\rangle \frac{\deg(\gamma)}{|T|}.
\end{aligned}
$$

Hence $\langle \Delta_B(P(f)), P^*(f^*)\rangle \geq 0$. Using Hölder's inequality, we have

$$
\begin{aligned}
& \langle P(f), P^*(f^*)\rangle \\
= & \langle P(f), f^*\rangle \\
= & \langle f + \delta^* f/2, f^*\rangle \\
= & \|f\|_M^p - \left\langle \sum_{\gamma\in K} f(\gamma)\frac{\deg(\gamma)}{|T|}, \sum_{\gamma''\in K} f^*(\gamma'')\frac{\deg(\gamma'')}{|T|}\right\rangle \\
\geq & \|f\|_M^p - \left\|\sum_{\gamma\in K} f(\gamma)\frac{\deg(\gamma)}{|T|}\right\| \left\|\sum_{\gamma''\in K} f^*(\gamma'')\frac{\deg(\gamma'')}{|T|}\right\|_{B^*} \\
\geq & \|f\|_M^p - \sum_{\gamma\in K} \|f(\gamma)\|\frac{\deg(\gamma)}{|T|} \sum_{\gamma''\in K} \|f^*(\gamma'')\|_{B^*}\frac{\deg(\gamma'')}{|T|} \\
\geq & \|f\|_M^p - \left(\sum_{\gamma\in K} \|f(\gamma)\|^p\frac{\deg(\gamma)}{|T|}\right)^{1/p} \left(\sum_{\gamma''\in K} \|f^*(\gamma'')\|_{B^*}^q\frac{\deg(\gamma'')}{|T|}\right)^{1/q} \\
= & \|f\|_M^p - \|f\|_M\|f^*\|_{M_*} = 0.
\end{aligned}
$$

Lemma 5.2.4. *Every $f \in M$ satisfies $\langle Df, D^*f^*\rangle = 2\langle \Delta_B f, f^*\rangle$.*

Proof. By the definition of D and D^*, we compute

$$
\begin{aligned}
\langle Df, D^*f^*\rangle &= \sum_{(\gamma,\gamma')\in T} \langle f(\gamma) - f(\gamma'), f^*(\gamma) - f^*(\gamma')\rangle \frac{1}{|T|} \\
&= 2\sum_{(\gamma,\gamma')\in T} \langle f(\gamma) - f(\gamma'), f^*(\gamma)\rangle \frac{1}{|T|} \\
&= 2\sum_{\gamma\in K}\left\langle \deg(\gamma)f(\gamma) - \sum_{(\gamma,\gamma')\in T} f(\gamma'), f^*(\gamma)\right\rangle \frac{1}{|T|} \\
&= 2\sum_{\gamma\in K}\left\langle f(\gamma) - \frac{1}{\deg(\gamma)}\sum_{(\gamma,\gamma')\in T} f(\gamma'), f^*(\gamma)\right\rangle \frac{\deg(\gamma)}{|T|} \\
&= 2\langle \Delta_B f, f^*\rangle.
\end{aligned}
$$

□

Proposition 5.2.5. *Every $f \in C^1$ satisfies*

$$\frac{1}{3}\langle d^1 f, \delta^1 f^* \rangle + \frac{\lambda_{B,p}(L(K))}{2}\langle \delta^* f, d^* f^* \rangle \geq (2\lambda_{B,p}(L(K)) - 1)\langle f, f^* \rangle.$$

Proof. By the definition of $\lambda_{B,p}(L(K))$ and Lemma 5.2.2, we have

$$\begin{aligned}
\langle \Delta_B(P(f)), P^*(f^*) \rangle &\geq \lambda_{B,p}(L(K))\langle P(f), P^*(f^*) \rangle \\
&= \lambda_{B,p}(L(K))\,\langle P(f), f^* \rangle \\
&= \lambda_{B,p}(L(K)) \left\langle f + \frac{\delta^* f}{2}, f^* \right\rangle \\
&= \lambda_{B,p}(L(K)) \left(\langle f, f^* \rangle + \sum_{\gamma \in K} \left\langle \frac{\delta^* f}{2}, f^*(\gamma) \right\rangle \frac{\deg(\gamma)}{|T|} \right) \\
&= \lambda_{B,p}(L(K)) \left(\langle f, f^* \rangle - \left\langle \frac{\delta^* f}{2}, \frac{d^* f^*}{2} \right\rangle \right) \\
&= \lambda_{B,p}(L(K))\,\langle f, f^* \rangle - \frac{\lambda_{B,p}(L(K))}{4}\langle \delta^* f, d^* f^* \rangle.
\end{aligned}$$

Using Lemma 5.2.4 and Proposition 5.2.1, we obtain

$$\begin{aligned}
& (2\lambda_{B,p}(L(K)) - 1)\langle f, f^* \rangle \\
= \;& 2\lambda_{B,p}(L(K))\langle f, f^* \rangle - \langle f, f^* \rangle \\
\leq \;& 2\left(\langle \Delta_B(P(f)), P^*(f^*) \rangle + \frac{\lambda_{B,p}(L(K))}{4}\langle \delta^* f, d^* f^* \rangle \right) - \langle f, f^* \rangle \\
= \;& \langle Df, D^* f^* \rangle + \frac{\lambda_{B,p}(L(K))}{2}\langle \delta^* f, d^* f^* \rangle - \langle f, f^* \rangle \\
= \;& \langle f, f^* \rangle + \frac{1}{3}\langle d^1 f, \delta^1 f^* \rangle + \frac{\lambda_{B,p}(L(K))}{2}\langle \delta^* f, d^* f^* \rangle - \langle f, f^* \rangle \\
= \;& \frac{1}{3}\langle d^1 f, \delta^1 f^* \rangle + \frac{\lambda_{B,p}(L(K))}{2}\langle \delta^* f, d^* f^* \rangle.
\end{aligned}$$

□

Theorem 6. *If $\lambda_{B/\tilde{B},p}(L(K)) > 1/2$ for every closed subspace $\tilde{B}$ of B, then Γ has Property (T_B).*

Proof. We should show that, for any non-trivial linear isometric representation π of Γ on B, there exists $C > 0$ such that $\max_{\gamma \in K} \|u - \pi'(\gamma)u\| \geq C\|u\|$ for all $u \in B' = B/B^{\pi(\Gamma)}$. Note that the representation π' of Γ on B' has no non-trivial vector. Applying Proposition 5.2.5 to B' and π', we obtain

$$\langle \delta^* f, d^* f^* \rangle \geq 2\left(2 - \frac{1}{\lambda_{B',p}(L(K)} \right) \langle f, f^* \rangle$$

for $f \in B^1$. By Proposition 5.1.8, if $\lambda_{B',p}(L(K)) > 1/2$, we obtain

$$\max_{\gamma \in K} \|u - \pi'(\gamma)u\| \geq (2 - 1/\lambda_{B',p}(L(K)))\|u\|$$

for all $u \in B'$. This completes the proof. $\square$

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